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Central Sequence Algebras of a Purely Infinite Simple C^*-algebra

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Abstract. We are concerned with a unital separable nuclear purely infinite simple C^*-algebra A satisfying UCT with a Rohlin flow, as a continuation of [12]. Our first result (which is independent of the Rohlin flow) is to characterize when two *central* projections in A are equivalent by a *central* partial isometry. Our second result shows that the K-theory of the central sequence algebra $A' \cap A^\omega$ (for an $\omega \in \beta\mathbf{N} \setminus \mathbf{N}$) and its *fixed point* algebra under the flow are the same (incorporating the previous result). We will also complete and supplement the characterization result of the Rohlin property for flows stated in [12].

1 Introduction

When A is a unital separable nuclear purely infinite simple C^*-algebra, Kirchberg and Phillips showed in [8] that $A' \cap A^\omega$ is purely infinite and simple, where A^ω is the ultrapower of A for an $\omega \in \beta\mathbf{N} \setminus \mathbf{N}$ (see the definition below). If α is a flow (or continuous action of $\mathbf{R}$) on A, α induces a non-continuous action of $\mathbf{R}$ on A^ω and we can take the α-continuous part A^ω_α of A^ω. When α has the Rohlin property, we have shown in [12] that the α-fixed point algebra $(A' \cap A^\omega_\alpha)^\alpha$ is again purely infinite and simple and the embedding $(A' \cap A^\omega_\alpha)^\alpha \subset A' \cap A^\omega$ induces an isomorphism $K_0((A' \cap A^\omega_\alpha)^\alpha) \cong K_0(A' \cap A^\omega)$. We will continue to study these objects. First we characterize when two projections in $A' \cap A^\omega$ (or hence in $(A' \cap A^\omega_\alpha)^\alpha$) are equivalent. Second we will show that the embedding $(A' \cap A^\omega_\alpha)^\alpha \subset A' \cap A^\omega$ also induces an isomorphism $K_1((A' \cap A^\omega_\alpha)^\alpha) \cong K_1(A' \cap A^\omega)$. Finally we will complete the proof of the main result of [12], which is an attempt to characterize the Rohlin property for flows. The result includes that α has the Rohlin property if and only if the crossed product $A \times_\alpha \mathbf{R}$ is purely infinite and simple and the dual flow $\hat{\alpha}$ has the Rohlin property. See 4.6 for details. We will also show that the trivial flow is obtained as a limit of cocycle perturbations of a Rohlin flow. In particular the Rohlin flow has a cocycle perturbation whose fixed point algebra contains the image of a unital endomorphism.

We recall ultrapowers of a C^*-algebra A. We denote by $\ell^\infty(A)$ the C^*-algebra of bounded sequences $x = (x_n)_{n=1}^\infty$ in A. For a free ultrafilter $\omega \in \beta\mathbf{N} \setminus \mathbf{N}$, we define

$$c^\omega(A) = \left\{ x \in \ell^\infty(A) \,\middle|\, \lim_{n\to\omega} \|x_n\| = 0 \right\},$$

which is a closed ideal of $\ell^\infty(A)$ and set $A^\omega = \ell^\infty(A)/c^\omega(A)$. We embed A into A^ω as constant sequences. It is known [8] that if A is a unital separable nuclear purely infinite simple C^*-algebra, then $A' \cap A^\omega$ is a unital purely infinite simple C^*-algebra. For

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each projection $e \in A^\omega$ we can choose a sequence (e_n) in $\mathcal{P}(A)$, the set of projections in A, such that (e_n) represents e, which will sometimes be denoted by $e = (e_n)$.

We denote by $\mathcal{U}(A)$ the group of unitaries of A (or $A+\mathbf{C}1$ if A is non-unital) and by $\mathcal{P}(A)$ the set of projections in A as above. If $e, p \in \mathcal{P}(A)$ almost commute with each other, then ep is close to a projection, whose (Murray-von Neumann) equivalence class is denoted by $[ep]_0$. If $e \in \mathcal{P}(A)$ almost commutes with $u \in \mathcal{U}(A)$, then $eu+1-e$ is close to a unitary, whose equivalence class (*i.e.,* homotopy class in $\mathcal{U}(A)$) is denoted by $[eu]_1$. Our first result, which is independent of flows, is as follows.

Corollary 1.1 *Let A be a unital separable nuclear purely infinite simple C^*-algebra satisfying the Universal Coefficient Theorem and let $\omega \in \beta\mathbf{N} \setminus \mathbf{N}$.*

Let $e_0, e_1 \in \mathcal{P}(A^\omega \cap A')$ and let $(e_{\sigma,n})$ be a sequence in $\mathcal{P}(A)$ representing e_σ. Then e_0 and e_1 are equivalent if and only if for any finite subsets $\mathcal{P} \subset \mathcal{P}(A)$ and $\mathcal{U} \subset \mathcal{U}(A)$ there is an $\Omega \in \omega$ such that for any $n \in \Omega$, it follows that $[e_{\sigma,n}, p] \approx 0$ and $[e_\sigma, u] \approx 0$ and

$$[e_{0,n}p]_0 = [e_{1,n}p]_0, \ [e_{0,n}u]_1 = [e_{1,n}u]_1$$

for all $p \in \mathcal{P}$ and $u \in \mathcal{U}$.

This will follow from Theorem 2.1 of Section 2.

If α is a flow on A, we can define an action $\overline{\alpha}$ of $\mathbf{R}$ on $\ell^\infty(A)$ by $t \mapsto \overline{\alpha}_t((x_n)) = (\alpha_t(x_n))$. We set $\ell^\infty_\alpha(A) = \{x \in \ell^\infty(A) \mid t \mapsto \overline{\alpha}_t(x) \text{ is continuous}\}$, which is the maximal C^*-subalgebra of $\ell^\infty(A)$ on which $\overline{\alpha}$ is strongly continuous. For an $\omega \in \beta\mathbf{N} \setminus \mathbf{N}$, we set $A^\omega_\alpha = \ell^\infty_\alpha(A) \big/ c^\omega(A) \cap \ell^\infty_\alpha(A)$. Note that $\overline{\alpha}$ induces a flow on A^ω_α, which we will denote by α. The flow α leaves $A' \cap A^\omega_\alpha$ invariant; the C^*-subalgebra of α-invariant elements there will be denoted by $(A' \cap A^\omega_\alpha)^\alpha$.

Corollary 1.2 *Let A be a unital separable nuclear purely infinite simple C^*-algebra satisfying the Universal Coefficient Theorem and let $\omega \in \beta\mathbf{N} \setminus \mathbf{N}$. Let α be a Rohlin flow on A. Then the embedding $(A' \cap A^\omega_\alpha)^\alpha \subset A' \cap A^\omega$ induces an isomorphism $K_*\left((A' \cap A^\omega_\alpha)^\alpha\right) \cong K_*(A' \cap A^\omega)$ for $* = 0, 1$.*

For $* = 0$ this is shown in [12]. The case for $* = 1$ will follow from 4.2 and 4.4.

2 Projections

We choose a small $\delta_0 > 0$ satisfying: If e, f are projections in the C^*-algebra A such that $\|[e, f]\| < \delta_0$ then $\chi_{[1/2,\infty)}(efe)$ defines a projection whose equivalence class is denoted by $[ef]_0$, where χ_C is the characteristic function of $C \subset \mathbf{R}$. Furthermore if $e \in \mathcal{P}(A)$ and $u \in \mathcal{U}(A)$ are such that $\|[e, u]\| < \delta_0$, then $ue + 1 - e$ is invertible, whose equivalence class is denoted by $[ue]_1$.

Theorem 2.1 *Let A be a separable nuclear purely infinite simple C^*-algebra satisfying the Universal Coefficient Theorem.*

For any finite subset $\mathcal{F}$ of A and $\epsilon > 0$, there exist a finite subset $\mathcal{P}$ of $\mathcal{P}(A)$, a finite subset $\mathcal{U}$ of $\mathcal{U}(A)$, a finite subset $\mathcal{G}$ of A, and $\delta \in (0, \delta_0)$ satisfying: For any pair e_0, e_1 in $\mathcal{P}(A) \setminus \{0\}$ such that

$$\|[e_\sigma, x]\| < \delta, \ x \in \mathcal{P} \cup \mathcal{U} \cup \mathcal{G}$$

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for $\sigma = 0, 1$ *and*

$$[pe_0]_0 = [pe_1]_0,\ p \in \mathcal{P},$$
$$[ue_0]_1 = [ue_1]_1,\ u \in \mathcal{U},$$

there is a partial isometry $v \in A$ *such that* $v^*v = e_0, vv^* = e_1$, *and*

$$\|[v, x]\| < \epsilon,\ x \in \mathcal{F}.$$

Remark 2.2 If $K_0(A)$ is finitely generated, we may take a fixed finite set $\{p_1, p_2, \dots, p_n\}$ for $\mathcal{P}$ in the above theorem, by enlarging $\mathcal{G}$ if necessary, such that $\{[p_1], [p_2], \dots, [p_n]\}$ generates $K_0(A)$. To see this we first note that any projection in A can be expressed in terms of $q \in \mathcal{P}(A)$ with $[q] = 0$ and $q \in \mathcal{P}(A)$ with $[q] = [p_i]$ for some i. If $[q] = 0$, then there are partial isometries $u, v \in A$ such that $u^*u = q = v^*v$ and $uu^* + vv^* = q$. Hence if $[e_\sigma, u] \approx 0$ and $[e_\sigma, v] \approx 0$, then it follows that $[e_\sigma q] = [e_\sigma uu^*] + [e_\sigma vv^*] = 2[e_\sigma q]$, *i.e.*, $[e_0 q] = 0 = [e_1 q]$. If $[q] = [p_i]$, then there is a partial isometry $u \in A$ such that $u^*u = q$ and $uu^* = p_i$. Hence if $[e_\sigma, u] \approx 0$, then $[e_\sigma q] = [e_\sigma p_i]$, *i.e.*, $[e_0 q] = [e_0 p_i] = [e_1 p_i] = [e_1 q]$. Thus, if $q \in \mathcal{P}(A)$ and if e_σ almost commutes with some finite set of elements associated with q as above, we can conclude that the equality $[e_0 q] = [e_1 q]$ follows from the conditions $[e_0 p_i] = [e_1 p_i]$ for $i = 1, \dots, n$. The same remark applies to $\mathcal{U}$.

Remark 2.3 We show that the conditions concerning $\mathcal{P}$ and $\mathcal{U}$ are necessary in the above theorem.

Assume that $K_0(A) = \mathbf{Z}$ and $[1_A] = 0$. Let e_0 and e_1 be non-zero projections in the Cuntz algebra $\mathcal{O}_\infty$ such that $[e_0] = 0$ and $[e_1] = 1$. Then $1_A \otimes e_\sigma$ is a projection in $A \otimes \mathcal{O}_\infty \cong A$ such that $[1_A \otimes e_\sigma] = 0$. If p is a projection in A such that $[p] = 1$, then

$$[p \otimes e_0] = 0,\ [p \otimes e_1] = 1.$$

This implies that if $v \in A \otimes \mathcal{O}_\infty$ satisfies that $v^*v = 1 \otimes e_0$ and $vv^* = 1 \otimes e_1$, then $\|[v, p \otimes 1]\| \geq 1$. Hence this shows that however central $1 \otimes e_\sigma$ is for $\sigma = 0, 1$, we cannot choose a partial isometry $v \in A \otimes \mathcal{O}_\infty$ with initial projection $1 \otimes e_0$ and final projection $1 \otimes e_1$, almost commuting with this particular p. The above assertion is shown as follows. If $\|[v, p \otimes 1]\| < 1$, then $\|v(p \otimes e_0)v^* - p \otimes e_1\| \leq \|[v, p \otimes 1](1 \otimes e_0)v^*\| = \|[v, p \otimes 1]\| < 1$, which implies that $p \otimes e_0$ and $p \otimes e_1$ are mutually equivalent, a contradiction.

Assume that $K_0(A) = 0$ and $K_1(A) = \mathbf{Z}$. Let e_0 and e_1 be non-zero projections in $\mathcal{O}_\infty$ such that $[e_0] = 0$ and $[e_1] = 1$. Then $1 \otimes e_\sigma$ is a projection in $A \otimes \mathcal{O}_\infty \cong A$ such that $[1 \otimes e_\sigma] = 0$. Let u be a unitary in A such that $[u] = 1$. Then $[u \otimes e_0] = 0$ and $[u \otimes e_1] = [u \otimes 1] = 1$. This implies that if $v \in A \otimes \mathcal{O}_\infty$ satisfies that $v^*v = 1 \otimes e_0$ and $vv^* = 1 \otimes e_1$, then $\|[v, u \otimes 1]\| \geq 2$. Because if $\|[v, u \otimes 1]\| < 2$, then $v^*(u \otimes e_1)v$ and $u \otimes e_0$ would be equivalent as unitaries in $(1 \otimes e_0)A \otimes \mathcal{O}_\infty(1 \otimes e_0)$, which is a contradiction.

By the uniqueness theorem proved in [8, 9] a unital separable nuclear purely infinite simple C^*-algebra with UCT is obtained as an inductive limit of finite direct

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sums of a C^*-algebra of the form $\mathcal{O} \otimes C^*(z)$, where $\mathcal{O}$ is a corner of a Cuntz algebra and $C^*(z)$ is the C^*-algebra generated by a unitary z with full spectrum (see [2]); we may further assume that the connecting maps are all injective. The above result is shown for (a corner of) the Cuntz algebra $\mathcal{O}_n$ with $n < \infty$ in [10, 3.5], where $\mathcal{P} = \{1\}$ and $\mathcal{U} = \emptyset$ suffice. The following lemma, as a generalization of this result, is a special case of the above theorem.

Lemma 2.4 *The above theorem is valid for a corner of a Cuntz algebra, where $\mathcal{U} = \emptyset$ suffices.*

Proof A corner of a Cuntz algebra can be given as $e(B \times_\alpha \mathbf{Z})e$, where B is a stable AF C^*-algebra with $K_0(B) \subset \mathbf{R}$, e is a projection in B, and α is an automorphism of B which does not preserve the trace τ, where τ is defined by $\tau(p) = [p]$ for $p \in \mathcal{P}(A)$. We may suppose that $\tau\alpha(e) < \tau(e)$.

We may suppose that there is an increasing sequence (B_n) of finite-dimensional C^*-subalgebras of B with dense union such that $\alpha(B_n) \subset B_{n+1}$, $B_n \subset \alpha(B_{n+1})$, $e \in B_1$, $\alpha(e) \in B_1$, $\alpha(e) \leq e$, $\alpha(e)$ has central support e in eB_1e, and any direct summand of B_n has a copy in any direct summand in B_{n+1} for any n. Note that $A = e(B \times_\alpha \mathbf{Z})e$ is a unital separable nuclear purely infinite simple C^*-algebra with $K_1(A) = 0$ [15]. Note also that α has the Rohlin property and is determined up to cocycle conjugacy by the number $\tau(\alpha(e))/\tau(e)$ [6, 3].

Let U denote the canonical unitary multiplier of $B \times_\alpha \mathbf{Z}$ implementing α and let $S = Ue \in A$. Then A is generated by the isometry S and the AF C^*-subalgebra eBe. We define an endomorphism λ of A by $\lambda(x) = SxS^*$, $x \in A$. Let $n \geq 2$. Since $A \cap (eB_ne)' = e(B \times_\alpha \mathbf{Z} \cap B_n')e$, we have, for any $x \in A \cap (eB_ne)'$, an $\hat{x} \in (B \times_\alpha \mathbf{Z}) \cap B_n'$ such that $\hat{x}e = x$, from which $U\hat{x}U^*\alpha(e) = \lambda(x)$. Since $U\hat{x}^*U^* \in B_{n-1}'$, we have that $\lambda(x) \in (A \cap (eB_{n-1}e)')\alpha(e)$. Thus, by using the fact that the multiplication by $\alpha(e)$ on $A \cap (eB_1e)'$ is an isomorphism and that $B_1 \subset \alpha(B_2)$, we define a unital homomorphism $\tilde{\lambda}$ of $A \cap B_2'$ into $A \cap B_1'$ by $\tilde{\lambda}(x)\alpha(e) = \lambda(x)$, where $A \cap B_n'$ should be understood as $A \cap (eB_ne)'$ with e the identity of A, or we should say we often use B_n to denote eB_ne if it is clear from the context. Note that $\tilde{\lambda}(A \cap B_n')$ is contained in $A \cap B_{n-1}'$ and contains $A \cap B_{n+1}'$. Since $\|[S, y]\| = \|SyS^* - y\alpha(e)\| = \|\tilde{\lambda}(y) - y\| = \|\tilde{\lambda}(y^*) - y^*\|$ for $y \in A \cap B_2'$, we have that $y \in A \cap B_2'$ almost commutes with S and S^* if and only if $\|\tilde{\lambda}(y) - y\| \approx 0$. In this way we may try to choose the desired v from $A \cap B_N'$ such that $\|\tilde{\lambda}(v) - v\| < \epsilon$ for any prescribed N and ϵ.

By the Rohlin property of α, we have, for any $N, n \in \mathbf{N}$ and $\epsilon' > 0$, a *Rohlin partition* $e_{10}, e_{11}, \ldots, e_{1,n-1}, e_{20}, e_{21}, \ldots, e_{2,n}$ of unity by projections in $e(B_M \cap B_N')e$ for a large $M > N$ such that

$$\max\left\{ \|\tilde{\lambda}(e_{\sigma,i}) - e_{\sigma,i+1}\| \,\middle|\, i = 0, 1, \ldots, n-3+\sigma,\ \sigma = 1, 2\right\} < \epsilon'.$$

We assume that N and n are sufficiently large and choose M as above.

Let $\{E_i;\ i = 1, 2, \ldots, K\}$ denote the set of minimal central projections in $eB_{M+2n+2}e$ and let p_i be a minimal projection in $E_iB_{M+2n+2}E_i$.

Let e_0, e_1 be non-zero projections in $A \cap B_{M+2n+2}'$ such that $\tilde{\lambda}(e_\sigma) \approx e_\sigma$ for $\sigma = 0, 1$ and $[e_0p_i]_0 = [e_1p_i]_0$ in $K_0(A)$ for $i = 1, 2, \ldots, K$. That is, we have set

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$\mathcal{P} = \{p_i \mid i = 1, 2, \ldots, K\}$. Let $\{F_j \mid j = 1, \ldots, K'\}$ denote the set of minimal central projections in $eB_{M+2n+1}e$. Since the condition $[e_0 p_i] = [e_1 p_i]$ implies that

$$[e_0 F_j] = [e_1 F_j] \text{ in } K_0\big(F_j(A \cap B'_{M+2n+1})F_j\big),$$

and since $e_\sigma F_j \neq 0$, we have a partial isometry $w \in A \cap B'_{M+2n+1}$ such that $w^* w = e_0$ and $ww^* = e_1$.

Since $\tilde{\lambda}(e_\sigma) \approx e_\sigma$, there is a $v_\sigma \in \mathcal{U}(A \cap B'_{M+2n})$ such that $v_\sigma \approx 1$ and $\mathrm{Ad}\ v_\sigma \tilde{\lambda}(e_\sigma) = e_\sigma$. Then $x = wv_0\tilde{\lambda}(w^*)v_1^*$ is a unitary in $e_1(A \cap B'_{M+2n})e_1$. We set $x_0 = e_1, x_1 = x$, and $x_k = x\,\mathrm{Ad}\ v_1\tilde{\lambda}(x_{k-1})$ for $k = 1, 2, \ldots$. Since $x_k \in e_1(A \cap B'_{M+2n+1-k})e_1$ and $K_1(e_1(A \cap B'_{M+n})e_1) = 0$, there is a rectifiable path w_k from e_1 to x_k in $\mathcal{U}(e_1(A \cap B'_{M+n})e_1)$ of length about π for $k = n, n+1$, *i.e.*, $w_k(0) = e_1$, $w_k(1) = x_k$, and $\|w_k(s) - w_k(t)\| < 2\pi|s - t|$ for $0 \le s < t \le 1$. By using those paths applied with $\tilde{\lambda}^{-k}$ with $k = 0, 1, \ldots, n$ and the Rohlin partition in $e(B_M \cap B'_N)e$, one defines a unitary $z \in e_1(A \cap B'_N)e_1$ such that $x = wv_0\tilde{\lambda}(w^*)v_1^* \approx z\tilde{\lambda}(z^*)$ (up to the order of $1/n$) [6]. More concretely we define

$$z = \sum_{k=0}^{n-1} x_{k+1}\tilde{\lambda}^{k-n+1}\left(w_n\left(k/(n-1)\right)^*\right) + \sum_{k=0}^{n} x_{k+1}\tilde{\lambda}^{k-n}\left(w_{n+1}(k/n)^*\right),$$

where we should note that $\tilde{\lambda}^{-1}$ maps $A \cap B'_m$ into $A \cap B'_{m-1}$. Then $w_1 = z^* w$ is a partial isometry in $A \cap B'_N$ such that $\tilde{\lambda}(w_1) \approx w_1$. Since $w_1^* w_1 = e_0$ and $w_1 w_1^* = e_1$, this concludes the proof. ■

Proof of Theorem 2.1 We may assume that A is unital by finding a projection E such that EAE almost contains $\mathcal{F}$ and by restricting everything to EAE.

As noted before (Lemma 2.4), we may assume that there is an increasing sequence (A_n) of unital C^*-subalgebras of A such that $A = \overline{\bigcup_n A_m}$, $A = \bigoplus_{k=1}^{K_n} A_{nk}$, and $A_{nk} = D_{nk} \otimes C^*(z_{nk})$, where D_{nk} is of the form $e(B \times_\alpha \mathbf{Z})e$ as in the previous lemma and z_{nk} is a unitary with full spectrum.

Let $\mathcal{F}$ be a finite subset of A and $\epsilon > 0$. We may suppose that $\mathcal{F}$ equals

$$\bigcup_{k=1}^{K_n} \mathcal{F}_{nk} \cup \{z_{nk}\}$$

for some n, where $\mathcal{F}_{nk} \subset D_{nk}$. We choose $\mathcal{P}_{nk} \subset \mathcal{P}(D_{nk})$, $\mathcal{G}_{nk} \subset D_{nk}$, and $\delta_{nk} > 0$ by applying 2.4 to D_{nk} with $(\mathcal{F}_{nk}, \epsilon)$.

Let E_{nk} denote the identity of A_{nk}. We approximate z_{nk} by

$$w \oplus w \oplus y^* \in \mathcal{U}(E_{nk}AE_{nk}),$$

where $[w] = [z_{nk}] = [y]$, $[w^* w] = [E_{nk}]$, $[y^* y] = -[E_{nk}]$, and $\mathrm{Spec}(w) = \mathbf{T} = \mathrm{Spec}(y)$ (in case $[z_{nk}] = 0$). Let v_{nk} be a self-adjoint unitary in $\mathcal{U}(E_{nk}AE_{nk})$ which switches the first two components of $w \oplus w \oplus y^*$ and is the identity on the support of the third.

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We approximate $0 \oplus w \oplus y^*$ by a unitary on $0 \oplus 1 \oplus 1$ with finite spectrum

$$\sum_{k=0}^{N-1} e^{2\pi ik/N} f_k$$

for a large N with $[f_k] = 0$ and $f_k \neq 0$ (*cf.* [12, 2.5]). We note that $F = \sum_{k=0}^{N-1} f_k$ satisfies that $F + vFv = 1 \oplus 1 \oplus 2$. We find a family $(f_{ij}^{(nk)})$ of matrix units $E_{nk}AE_{nk}$ such that $f_{jj}^{(nk)} = f_j$. We set

$$\mathcal{P} = \bigcup_{k=1}^{K_n} \mathcal{P}_{nk},$$

$$\mathcal{U} = \bigcup_{k=1}^{K_n} \{z_{nk}p + 1 - p \mid p \in \mathcal{P}_{nk}\},$$

$$\mathcal{G} = \bigcup_{k=1}^{K_n} \mathcal{G}_{nk} \cup \{E_{nk}, f_{ij}^{(nk)}, v_{nk}\}.$$

We will take a sufficiently small $\delta > 0$.

Let e_0, e_1 be a pair in $\mathcal{P}(A) \setminus \{0\}$ such that

$$\|[e_\sigma, x]\| < \delta,\ x \in \mathcal{P} \cup \mathcal{U} \cup \mathcal{G}$$

and

$$[pe_0]_0 = [pe_1]_0,\ p \in \mathcal{P},$$

$$[ue_0]_1 = [ue_1]_1,\ u \in \mathcal{U}.$$

Since e_σ almost commutes with E_{nk}, we can discuss the pairs e_0E_{nk} and e_1E_{nk} in $E_{nk}AE_{nk}$ separately. Thus we have the following situation: $e(B \times_\alpha \mathbf{Z})e \otimes C^*(z)$ is a unital C^*-subalgebra of A, where (B, α) is as described as in the proof of Lemma 2.4, and the two non-zero projections $e_0, e_1 \in \mathcal{P}(A \cap B_m')$ are equivalent in $A \cap B_m'$ for a sufficiently large m, and satisfy

$$[e_\sigma, z] \approx 0, \quad [e_\sigma, f_{ij}] \approx 0, \quad [e_\sigma, v] \approx 0, \quad \tilde{\lambda}(e_\sigma) \approx e_\sigma,$$

$$[zpe_0]_1 = [zpe_1]_1,\ p \in \mathcal{P},$$

where we have used the notation in the proof of 2.4. In particular, $\mathcal{P}$ is the set of minimal projections each of which is chosen from a direct summand of eB_me. From the last condition it follows that $[ze_0]_1 = [ze_1]_1$ in $K_1(A \cap B_{m-1}')$. The second and third conditions imply that even if $[ze_\sigma]_1 = 0$, the spectrum of (a unitary in $e_\sigma Ae_\sigma$ close to) ze_σ is almost dense in $\mathbf{T}$ (because $e_\sigma F \neq 0$ or $e_\sigma vFv^* \neq 0$ where $F = \sum_k f_k$). Hence it follows [5] that there is a $w \in A \cap B_{m-1}'$ such that $w^*w = e_0$, $ww^* = e_1$, and

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$wze_0w^* \approx ze_1$. Note also that if $e_\sigma \neq 1$, then the spectrum of $z(1-e_\sigma)$ is also almost dense in $\mathbf{T}$.

We make another assumption on the choice of the increasing sequence (B_m) of finite-dimensional C^*-subalgebras of B: For any $m = 1, 2, \ldots$ there is a $v \in \mathcal{U}(e(B_{m+1} \cap B_m')e)$ such that $vS \in A \cap (eB_me)'$ and that for any $p \in \mathcal{P}(e(B_{m+1} \cap B_m')e)$ with $p \leq vSS^*v^*$ the projection $q = (vS)^*p(vS) \in e(B_{m+2} \cap B_m')e$ satisfies that $[q] \geq [p]$ in $K_0(e(B_{m+2} \cap B_m')e)$. We can see that this does not cause the loss of generality as follows. Let $\{E_{mk} \mid k = 1, 2, \ldots, K_m\}$ denote the set of minimal central projections of eB_me. By passing to a subsequence, we may suppose that $\alpha(E_{mk}) = SE_{mk}S^*$ is equivalent to a subprojection of E_{mk} in $eB_{m+1}e$. Then there is a $v \in \mathcal{U}(eB_{m+1}e)$ such that $p_k = vSE_{mk}S^*v^* \leq E_{mk}$ for any k. Note that $E_{m+1,\ell}p_k$ is a projection in $E_{m+1,\ell}E_{mk}eB_{m+1}eE_{mk}$ (which is a full matrix algebra) and has dimension divisible by $[m,k]$, where $[m,k]$ is given by $E_{mk}eB_me \cong M_{[m,k]}$. Hence, by changing v if necessary, we may suppose that $E_{m+1,\ell}p_k \in E_{m+1,\ell}E_{mk}eB_{m+1}eE_{mk} \cap (E_{m+1,\ell}E_{mk}eB_me)'$ for any ℓ, which says that

$$p_k = vSE_{mk}S^*v^* \in E_{mk}eB_{m+1}eE_{mk} \cap (E_{mk}eB_me)'.$$

Define a homomorphism $\phi_k\colon p_kE_{mk}eB_me \to p_kE_{m+1,\ell}E_{mk}eB_{m+1}eE_{mk}p_k$ by

$$\phi_k(p_kx) = E_{m+1,\ell}vSxS^*v^*,\ x \in E_{mk}eB_me.$$

Since this is a unital isomorphism of a full matrix algebra into a full matrix algebra, this must be unitarily equivalent to the inclusion

$$p_kE_{mk}eB_me \subset p_kE_{m+1,\ell}E_{mk}eB_{m+1}eE_{mk}p_k.$$

Hence there is a unitary w_k in $p_kE_{mk}eB_{m+1}eE_{mk}p_k$ such that

$$w_kvSxS^*v^*w_k^* = p_kx,\ x \in E_{mk}eB_me.$$

Let $w = \sum_k w_k + (e - \sum_k p_k)$ and replace v by $wv \in \mathcal{U}(eB_{m+1}e)$. Then it follows that $p_k = vSE_{mk}S^*v^*$ and $vSxS^*v^* = vSS^*v^*x = xvSS^*v^*$ for $x \in eB_me$. The latter condition implies that $[x, vS] = 0$, $x \in eB_me$, *i.e.*, $vS \in A \cap (eB_me)'$. The other condition can be met by passing to a subsequence if necessary.

We shall show first that there is no loss of generality to assume that $e_0e_1 = 0$.

If $e_0 = 1 = e_1$, then there is nothing to prove in the first place. Hence suppose that $e_1 \neq 1$. Since $\tilde{\lambda}(e_\sigma) \approx e_\sigma$ and $e_\sigma, \tilde{\lambda}(e_\sigma) \in A \cap B_{m-1}'$, there is a $v_\sigma \in \mathcal{U}(A \cap B_{m-1}')$ for $\sigma = 0, 1$ such that $v_\sigma \approx 1$ and $\operatorname{Ad} v_\sigma\tilde{\lambda}(e_\sigma) = e_\sigma$. By using the Rohlin property for α on B, we get a Rohlin partition of unity $\{p_{10}, p_{11}, \ldots, p_{1,n-1}, p_{20}, \ldots, p_{2n}\}$ by projections in $e(B_{m-2} \cap B_{\ell+1}')e$ for $n \gg 1$ and $m \gg \ell \gg 1$ such that $\tilde{\lambda}(p_{\sigma,i}) \approx p_{\sigma,i+1}$. (We actually choose ℓ first and then m to accommodate such a Rohlin partition.) We find a $v_2 \in \mathcal{U}(e(B_{m-1} \cap B_\ell')e)$ such that $v_2 \approx 1$ and $\operatorname{Ad} v_2\tilde{\lambda}(p_{\sigma,i}) = p_{\sigma,i+1}$. Since the spectrum of $z(1-e_1)p_{\sigma,i}$ is independent of i (since it is left invariant under $\operatorname{Ad}(v_2v_1)\tilde{\lambda}$), it follows that the spectrum of $z(1-e_1)p_0$ is almost dense in $\mathbf{T}$ for $p_0 =$

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$p_{10} + p_{20}$. We then find a partial isometry $w \in A \cap B'_\ell$ such that $w^*w = e_0$, $ww^* \le (1 - e_1)p_0$, and $[z, w] \approx 0$ (see [5]). We define

$$W = n^{-1/2} \sum_{k=0}^{n-1} (L_{v_2 v_1} R_{v_0^*} \tilde{\lambda})^k (w),$$

which is a partial isometry in $A \cap B'_{\ell-n}$ such that $W^*W = e_0$, $WW^* \le 1 - e_1$, $[z, W] \approx 0$, and $\tilde{\lambda}(W) \approx W$ (up to $n^{-1/2}$). Here L_x (resp. R_x) denotes the bounded operator on A defined by $L_x y = xy$ (resp. $R_x y = yx$.) Note that $e_0' = WW^*$ is connected with e_0 by the partial isometry W which commutes with elements from a prescribed finite subset. Hence the pair e_0 and e_0' (as well as e_1) should satisfy the same kind of conditions in the statement (if we start with stronger conditions imposed on the pair (e_0, e_1).) Thus we are left with the two projections e_0' and e_1 which are mutually orthogonal and can be chosen to have prescribed properties.

Now we assume that $e_0 e_1 = 0$. We choose $v \in \mathcal{U}(A \cap B'_{m-1})$ such that $v \approx 1$ and $\mathrm{Ad}\ v\tilde{\lambda}(e_\sigma) = e_\sigma$. Note that we have chosen $w \in A \cap B'_{m-1}$ such that $w^*w = e_0$, $ww^* = e_1$, and $[w, z] \approx 0$. Then $x = wv\tilde{\lambda}(w^*)v^*$ is a unitary in $e_1(A \cap B'_{m-2})e_1$. Moreover since $\tilde{\lambda}(z) = z$, x almost commutes with ze_1. We set $x_0 = e_1$, $x_1 = x$, and $x_k = x\,\mathrm{Ad}\ v_1\tilde{\lambda}(x_{k-1})$ for $k = 2, 3, \ldots, n+1$. We may suppose that $[x_k, z] \approx 0$ for k up to $n+1$. By the following lemma 2.5 we have that $[x_k]_1 = 0$ in $K_1(e_1(A \cap B'_{m-n-2})e_1)$ and the Bott element $B(x_k, ze_1)$ is 0 in $K_0(e_1(A \cap B'_{m-n-2})e_1)$ for $k \le n+1$ (see [13, 7]). By 8.1 of [1] we have a rectifiable path (of length less than $5\pi + 1$) from x_k to e_1 in $\mathcal{U}(e_1Ae_1 \cap B'_{m-n-1})$ almost commuting with ze_1 for $k = n, n+1$. By using these paths (applied by $\tilde{\lambda}^{-k}$ with k up to n) and the Rohlin partition in $e(B_{m-2n-2} \cap B'_N)e$ (with $m - 2n - 2 \gg N$), we will obtain $\zeta \in \mathcal{U}(A \cap B'_N)$ such that $x \approx \zeta\tilde{\lambda}(\zeta^*)$. Then ζ^*w will be the desired isometry just as in the proof of Lemma 2.4. See also [12] for a similar proof.

Lemma 2.5 *With w, e_0, e_1, z, v as above,*

$$[wv\tilde{\lambda}(w^*)v^*] = 0$$

in $K_1(e_1Ae_1 \cap (e_1B_{m-3}e_1)')$ and

$$B(wv\tilde{\lambda}(w^*)v^*, ze_1) = 0$$

in $K_0(e_1Ae_1 \cap (e_1B_{m-3}e_1)')$. Moreover, with $x_1 = wv\tilde{\lambda}(w^*)v^*$, and x_k, $k = 2, 3, \ldots, n+1$, as above, $[x_k] = 0$ in $K_1(e_1Ae_1 \cap (e_1B_{m-2-k}e_1)')$ and $B(x_k, ze_1) = 0$ in $K_0(e_1Ae_1 \cap (e_1B_{m-2-k}e_1)')$.

To prove this lemma we prepare a couple of lemmas. We denote by $\mathcal{I}(A)$ the set of non-unitary isometries of A. When $z \in \mathcal{U}(A)$ and $p \in \mathcal{P}(A)$ almost commute, $[zp]$ is the equivalence class of a unitary close to $zp + 1 - p$ and $\mathrm{Spec}(zp)$ is the spectrum of such a unitary and is defined only up to the order $\|[z, p]\|$ (if $[zp]_1 = 0$).

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Lemma 2.6 *Let $s_0, s_1 \in \mathcal{I}(A)$ and $z \in \mathcal{U}(A)$ such that $[s_\sigma, z] \approx 0$ and $\mathrm{Spec}(z(1 - s_\sigma s_\sigma^*))$ is almost dense in $\mathbf{T}$ for $\sigma = 0, 1$. Then there is a rectifiable path s in $\mathcal{I}(A)$ such that $s(0) = s_0$, $s(1) = s_1$, and $[s(t), z] \approx 0$.*

Proof Since $[z(1 - s_\sigma s_\sigma^*)] = 0$, it follows that there is a partial isometry v such that $v^*v = 1 - s_0 s_0^*$, $vv^* = 1 - s_1 s_1^*$, and $[z, v] \approx 0$. Then the unitary $u_1 = s_1 s_0^* + v$ satisfies that $u_1 s_0 = s_1$ and $[u_1, z] \approx 0$. We may suppose that $[u_1] = 0$ and $B(u_1, z) = 0$ by modifying v if necessary. (There is a $v' \in \mathcal{U}(A)$ such that $v' = v'(1 - s_0 s_0^*) + s_0 s_0^*$, $(v' - 1)z \approx v' - 1$, and $[v']$ is an arbitrary element of $K_1(A)$. There is another $v'' \in \mathcal{U}(A)$ such that $v'' = v''(1 - s_0 s_0^*) + s_0 s_0^*$, $[v''] = 0$, $[v'', z] \approx 0$, and $B(v'', z)$ is an arbitrary element of $K_0(A)$.) Then there is a rectifiable path u such that $u(0) = 1$, $u(1) = u_1$, and $[u(t), z] \approx 0$ (see [1]). Hence the path $s(t) = u(t)s_0$ satisfies that $s(0) = s_0$, $s(1) = s_1$, and $[s(t), z] \approx 0$. ■

Lemma 2.7 *Let D be a finite-dimensional C^*-subalgebra of A and let $s_0, v_0 \in \mathcal{I}(A \cap D')$ and $z \in \mathcal{U}(A \cap D')$ such that $[s_0, z] \approx 0$, $[v_0, z] \approx 0$, $s_0 s_0^* + v_0 v_0^* \leq 1$, and $\mathrm{Spec}(zp)$ is almost dense for each minimal central projection p of D. Then there is a continuous map s of $[0, \infty)$ into $\mathcal{I}(A \cap D')$ such that $s(0) = s_0$, $[s(t), z] \approx 0$, and $\lim_{t \to \infty} \|[s(t), x]\| = 0$ for $x \in A$. Moreover there is a continuous path v of $[0, \infty)$ into $\mathcal{I}(A \cap D')$ such that $v(0) = v_0$, $[v(t), z] \approx 0$, and $v(t)v(t)^* \leq 1 - s(t)s(t)^*$.*

Proof Let $s_1 \in \mathcal{I}(\mathcal{O}_\infty)$, where $\mathcal{O}_\infty$ is the Cuntz algebra generated by infinitely many isometries. There is a continuous map f of $[0, 1]$ into $\mathcal{I}(\mathcal{O}_\infty \otimes \mathcal{O}_\infty)$ such that $f(0) = s_1 \otimes 1$ and $f(1) = 1 \otimes s_1$. We regard f as a map of $[0, 1]$ into $\mathcal{O}_\infty \otimes \mathcal{O}_\infty \otimes 1 \otimes 1 \cdots \subset \bigotimes_0^\infty \mathcal{O}_\infty$. Let γ denote the one-sided shift on $E = \bigotimes_0^\infty \mathcal{O}_\infty$ and define a continuous map s of $[0, \infty)$ into $\mathcal{I}(E)$ by

$$s(t) = \gamma^n(f(t - n)), \quad t \in [n, n + 1).$$

It follows that $s(0) = s_1 \otimes 1 \otimes 1 \cdots$ and $\lim \|[s(t), x]\| = 0$ for $x \in E$. Note, by the proof of the previous lemma, that there is a continuous path u in $\mathcal{U}(E)$ such that $s(t) = u(t)s_1$. Note by [8] that $E \cong \mathcal{O}_\infty$.

Since $A \cong A \otimes \mathcal{O}_\infty$, we may identify A with $A \otimes \mathcal{O}_\infty$ and assume that $s_0, z \in A \otimes 1$ and $D \subset A \otimes 1$ (by modifying them slightly). Since we have constructed a continuous map s of $[0, \infty)$ into $\mathcal{I}(1 \otimes \mathcal{O}_\infty)$ such that $\lim \|[s(t), x]\| = 0$ for $x \in 1 \otimes \mathcal{O}_\infty$, it suffices to find a path connecting s_0 and $s(0)$ in $\mathcal{I}(A \otimes \mathcal{O}_\infty \cap D')$ almost commuting with z. Since $A \otimes \mathcal{O}_\infty \cap D'$ is a finite direct sum of C^*-algebras like A, this follows from the previous lemma if the condition on the spectrum of $zp(1 - s_0 s_0^*)$ is met for each minimal central projection p of D. (The condition for $zp(1 - s(0)s(0)^*)$ is obviously satisfied.) Since $1 - s_0 s_0^* \geq v_0 v_0^*$ and $zpv_0 v_0^* \approx v_0 zpv_0^*$, we have that $\mathrm{Spec}(zp(1 - s_0 s_0^*))$ almost contains $\mathrm{Spec}(zp)$, which is almost dense. Thus we can apply the previous lemma as asserted.

Note that the path s is defined as $s(t) = u(t)s_0$ with a path u in $\mathcal{U}(A \cap D')$ such that $u(0) = 1$ and $[u(s), z] \approx 0$ uniformly in $s \in [0, \infty)$. Hence the last part follows by defining $v(t) = u(t)v_0$. ■

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Lemma 2.8 *Let $u, v \in \mathcal{U}(A \otimes C[0,1])$ be such that $u(0) = v(0)$ and* $\mathrm{Spec}(u(t)) = \mathbf{T} = \mathrm{Spec}(v(t))$. *Then for any $\epsilon > 0$ there is a $\zeta \in \mathcal{U}(A \otimes C[0,1])$ such that $\zeta(0) = 1$ and $\|\zeta u \zeta^* - v\| < \epsilon$.*

Proof We take a large integer N such that $1/N < \epsilon$. We approximate u by a unitary $u_1 \oplus u'$ up to the order of ϵ, where the unitary u' has spectrum $\{\omega \in \mathbf{C} \mid \omega^N = 1\}$ and is given by

$$u' = \sum_{k=0}^{N-1} e^{2\pi i k/N} e_k.$$

We assume that $\sum_k [e_k] = 2[1]$ (and so $[u_1^* u_1] = -[1]$). We approximate v by a unitary $v' \oplus v_1$ up to the order of ϵ, where

$$v' = \sum_{k=0}^{N-1} e^{2\pi i k/N} p_k$$

with $p_k \neq 0$ and $[p_k] = 0$, which entails that $[v_1^* v_1] = [1]$. We then approximate u' by $s_1 v_1^* s_1^* \oplus s_2 v_1 s_2^*$, where s_1, s_2 are partial isometries such that $s_1 s_1^* + s_2 s_2^* = u' u'^*$ and $s_1^* s_1 = s_2^* s_2 = v_1 v_1^*$. Since $u_1 \oplus s_1 v_1^* s_1^*$ has trivial K_1 and spectrum $\mathbf{T}$, we can approximate it by a unitary u'', which is given by

$$u'' = \sum_{k=0}^{N-1} e^{2\pi i k/N} q_k$$

with $q_k \neq 0$ and $[q_k] = 0$. We find a partial isometry $y \in A$ such that $y q_k y^* = p_k$ and $y^* y = \sum_k q_k$. Since $u \approx u_1 \oplus u' \approx u_1 \oplus s_1 v_1^* s_1^* \oplus s_2 v_1 s_2^* \approx u'' \oplus s_2 v_1 s_2^*$ and $v \approx v' \oplus v_1$, and since the unitary $\zeta = y + s_2^*$ satisfies that $\zeta(u'' \oplus s_2 v_1 s_2^*)\zeta = v' \oplus v_1$, it follows that that $\|\zeta u \zeta^* - v\|$ is of the order of ϵ.

Note that $\zeta(0)$ may not be 1. If the Bott element $B(\zeta(0), u(0))$ vanishes, there is a continuous path $z(t)$ such that $z(0) = 1$, $z(1) = \zeta(0)$, and $[z(t), u(0)] \approx 0$ (see [1]). Hence in this case we can modify $\zeta(t)$ around $t = 0$ so that $\zeta(0) = 1$, retaining the condition that $\zeta(t)u(t)\zeta(t)^* \approx v(t)$ for t near 0, where $u(t) \approx u(0) \approx v(t)$.

If $B(\zeta(0), u(0)) \neq 0$, then we find a $\eta \in \mathcal{U}(A \otimes C[0,1])$ such that

$$[\eta, u] \approx 0 \quad \text{and} \quad B(\eta(t), u(t)) = -B(\zeta(0), u(0)).$$

Then it would follow that $(\zeta\eta)u(\zeta\eta)^* \approx v$ and $B(\zeta(0)\eta(0), u(0)) = 0$, which would produce the desired unitary by modifying $\zeta\eta$. We can get such an η as follows. We approximate u by $u_1 \oplus u'$ as above, where this time u' should be $\sum_k e^{2\pi i k/N} e_k$ with $[e_k] = B(\zeta(0), u(0))$. Then we find an $\eta \in \mathcal{U}(A \otimes C[0,1])$ such that $\eta e_k \eta^* = e_{k+1}$ with $e_N = e_0$ and $\eta(1 - \sum e_k) = 1 - \sum_k e_k$. This η satisfies the required condition (see 4.1 and 8.1 of [1]). ■

Proof of Lemma 2.5 We have supposed that $e_0, e_1 \in A \cap B_m'$ ($= A \cap (eB_m e)'$ more precisely) and $e_0 e_1 = 0$ and chosen a $v \in \mathcal{U}(A \cap B_{m-1}')$ such that $v \approx 1$ and

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$\mathrm{Ad}\, v\tilde{\lambda}(e_\sigma) = e_\sigma$, *i.e.*, $S \approx vS \in A \cap \{e_0, e_1\}'$. Note that $(vS)(vS)^* = \alpha(e)$. By the assumption there is a $u \in \mathcal{U}(eB_{m-2}e)$ such that $uvS \in A \cap B'_{m-3} \cap \{e_0, e_1\}'$ and $p = uvS(uvS)^* \in e(B_{m-2} \cap B'_{m-3})e$. We have chosen $w \in A \cap B'_{m-1}$ such that $w^*w = e_0$, $ww^* = e_1$, and $[w, z] \approx 0$. Since $x = wv\tilde{\lambda}(w^*)v^*$ is a unitary in $e_1(A \cap B'_{m-2})e_1$ and $[u, e_1] = 0$, we have that $x = uxu^*$.

Let $s_0 = uvS$ and note that $[s_0, z] \approx 0$. We may suppose that $2[\alpha(e)] < [e]$ in $K_0(B_1)$ in the first place and that $2[p] < [e]$ in $K_0(e(B_{m-2} \cap B'_{m-3})e)$. Thus we may suppose that there is an isometry $b_0 \in A$ (of the form bs_0 with some $b \in e(B_{m-2} \cap B'_{m-3})e$) such that $b_0b_0^* \in e(B_{m-2} \cap B'_{m-3})e$ such that $[b_0, z] \approx 0$, $[b_0, e_\sigma] = 0$, and $s_0s_0^* + b_0b_0^* \leq e$.

Let s be a continuous path in $\mathcal{I}(A \cap B'_{m-3} \cap \{e_0, e_1\}')$ such that $s(0) = s_0 = uvS$, $[s(t), z] \approx 0$, and $\lim \|[s(t), x]\| = 0$ for $x \in A$. Note that there is another path b in $\mathcal{I}(A \cap B'_{m-3} \cap \{e_0, e_1\}')$ such that $[b(t), z] \approx 0$, $b(0) = b_0$, and $s(t)s(t)^* + b(t)b(t)^* \leq 1$. Let $p(t) = s(t)s(t)^*$ and $q(t) = wp(t)w^*$, which are continuous paths in $\mathcal{P}(A \cap B'_{m-3} \cap \{e_0, e_1\}')$. Note that $\|q(t) - p(t)e_1\| \to 0$ as $t \to \infty$. We will assert that there is a continuous path v in $\mathcal{U}(e_1(A \cap B'_{m-3})e_1)$ such that

$$v(0) = e_1, \quad v(t)q(t)v(t)^* = p(t)e_1,$$
$$\lim_{t\to\infty} v(t) \text{ exists}, \quad [v(t), z] \approx 0.$$

If this is shown, then $U(t) = s(t)^*v(t)ws(t)w^*v(t)^*$ is a unitary in $e_1(A \cap B'_{m-3})e_1$, because $ws(t)w^*v(t)^* \cdot v(t)ws(t)^*w^* = q(t)$ and

$$U(t)U(t)^* = s(t)^*v(t)q(t)v(t)^*s(t) = e_1,$$

etc. Note also that $U(0) = (uvS)^*w(uvS)w$, $\lim_{t\to\infty} U(t) = e_1$, and $[U(t), z] \approx 0$. Hence $t \mapsto s(t)^*v(t)ws(t)w^*v(t)^*$ is a continuous path in $\mathcal{U}(e_1(A \cap B'_{m-3})e_1)$, almost commuting with z, from $(uvS)^*w(uvS)w$ to e_1. Since $xp = w(uvS)w^*(uvS)^*$, $[uvS, z] \approx 0$, and $[xp + e_1(1-p)] = [(uvS)^*w(uvS)w^*]$ in $K_1(e_1(A \cap B'_{m-3})e_1)$, this implies the assertions for $xp + e_1(1-p)$.

We shall show the above assertion on v. Let f be a minimal central projection of $eB_{m-3}e$. Since $zfs(t)s(t)^*e_\sigma \approx s(t)zfe_\sigma s(t)^*$, we have that $[z, fp(t)e_\sigma] \approx 0$ and $\mathrm{Spec}(zfp(t)e_\sigma)$ is almost dense in $\mathbf{T}$. Since $zfwp(t)w^* \approx wzfp(t)e_0w^*$, we have that $[z, fq(t)] \approx 0$ and $\mathrm{Spec}(zfq(t))$ is almost dense in $\mathbf{T}$. Since $1 - p(t) \geq b(t)b(t)^*$ and $zfe_\sigma b(t)b(t)^* \approx b(t)zfe_\sigma b(t)^*$, we have that $\mathrm{Spec}(zfe_\sigma(1 - p(t)))$ is almost dense. Since $zf(e_1 - q(t)) = zfw(1 - p(t))w^* \approx wzf(1 - p(t))w^*$, we have that $\mathrm{Spec}(zf(e_1 - q(t))$ is almost dense.

Since $q(0) = wp(0)w^* = p(0)ww^* = p(0)e_1$ and $q(t) \leq e_1$, there is a path y in $\mathcal{U}(e_1(A \cap B'_{m-3})e_1)$ such that $y(0) = 1$ and

$$y(t)q(t)y(t)^* = p(t)e_1.$$

There is again a path η in $\mathcal{U}(e_1(A \cap B'_{m-3})e_1)$ such that $\eta(0) = 1$ and

$$\eta(t)p(t)e_1\eta(t)^* = p(0)e_1.$$

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Then we compare the paths

$$t \mapsto \mathrm{Ad}(\eta(t)y(t))(zq(t)) \text{ and } t \mapsto \mathrm{Ad}(\eta(t))(zp(t)e_1)$$

in the unitary group of $p(0)e_1(A \cap B'_{m-3})p(0)e_1$ and also the paths

$$t \mapsto \mathrm{Ad}\left(\eta(t)y(t)\right)\left(z(e_1 - q(t)\right) \text{ and } t \mapsto \mathrm{Ad}\left(\eta(t)\right)\left(ze_1(1 - p(t))\right)$$

in the unitary group of $(1 - p(0))e_1(A \cap B'_{m-3})e_1(1 - p(0))$. Let T be so large that $q(t) \approx p(t)e_1$ for all $t \geq T$. By using the density of the spectra of these unitaries in each direct summands, we apply the previous lemma to find a path ζ in $\mathcal{U}(e_1(A \cap B'_{m-3})e_1)$ such that $[\zeta(t), p(0)e_1] = 0$ and

$$\mathrm{Ad}\left(\zeta(t)\eta(t)y(t)\right)(ze_1) \approx \mathrm{Ad}\ \eta(t)(ze_1) \text{ for } t \in [0, T].$$

Let $v(t) = \eta(t)^*\zeta(t)\eta(t)y(t)$ for $t \in [0, T]$. Then $v(t), t \in [0, T]$ is a path in $\mathcal{U}(e_1(A \cap B'_{m-3})e_1)$ satisfying that $v(t)q(t)v(t)^* = p(t)e_1$ and $[v(t), z] \approx 0$. We can extend $v(t)$ for $t \geq T$ in a small vicinity of $v(T)$ retaining these conditions. (For example we can use the polar decomposition of

$$p(t)e_1v(T)q(t)v(T)^* + e_1(1 - p(t))v(T)(e_1 - q(t))v(T)^*,$$

which is close to e_1 for $t \geq T$, to modify $v(T)$.) We may further suppose that $\lim_{t\to\infty} v(t)$ exists (*e.g.*, by repeating the above modifications for larger T). This concludes the proof of the assertion on v.

There are a finite number of partial isometries $\{y_i \mid i = 1, \dots, K\}$ in $e(B_{m-2} \cap B'_{m-3})e$ such that $y_i = (e - p)y_ip$ and $\sum_k y_iy_i^* = e - p$. Let $y_0 = p$. Then $x = \sum_{i=0}^K y_ixy_i^* = \sum_{i=0}^K y_iw(uvS)w^*(uvS)^*y_i^*$. With $p_i = y_i^*y_i \in eB_{m-2}e \cap (eB_{m-3}e)'$, we have that

$$\begin{aligned}[y_iw(uvS)w^*(uvS)^*y_i^*] &= [p_iw(uvS)w^*(uvS)^*p_i] \\ &= [(uvS)^*p_iw(uvS)w^*(uvS)^*p_i(uvS)]\end{aligned}$$

in $K_1(A \cap B'_{m-3})$. Since $q_i = (uvS)^*p_i(uvS) \in eB_{m-1}e \cap (eB_{m-3}e)'$ and $[q_i] \geq [p_i]$ in $K_0(eB_{m-1}e \cap (eB_{m-3}e)')$, we may suppose that $q_i \geq p_i$ by modifying u using a unitary in $eB_{m-1}e \cap (eB_{m-3}e)'$. There is a continuous path s_i in $\mathcal{I}(q_i(A \cap B'_{m-3} \cap \{e_0, e_1\}')q_i)$ such that $s_i(0) = p_iuvS$, $[s_i(t), zq_i] \approx 0$, and $\lim_{t\to\infty} \|[s_i(t), x]\| = 0$ for $x \in q_iAq_i$. Comparing the paths $t \mapsto s_i(t)s_i(t)^*e_1$ and $t \mapsto ws_i(t)s_i(t)^*w^*$ in $\mathcal{P}(e_1q_i(A \cap B'_{m-3})q_ie_1)$ with $s_i(0)s_i(0)^*e_1 = p_i\alpha(e)e_1 = ws_i(0)s_i(0)^*w^*$, we assert, as before, that there is a continuous path v_i in $\mathcal{U}(e_1q_i(A \cap B'_{m-3})q_ie_1)$ such that $v_i(0) = e_1q_i$, $v_i(t)ws_i(t)s_i(t)^*w^*v_i(t)^* = s_i(t)s_i(t)^*e_1$, $\lim_{t\to\infty} v_i(t)$ exists, and $[v_i(t), ze_1q_i] \approx 0$.

Let $U_i(t) = s_i(t)^*v_i(t)ws_i(t)w^*v_i(t)^*$, which is a unitary in $e_1q_i(A \cap B'_{m-3})q_ie_1$. This is because $w^*v_i(t)^*s_i(t) \cdot s_i(t)^*v_i(t)w = s_i(t)s_i(t)^*e_0$ and

$$U_i(t)^*U_i(t) = v_i(t)ws_i(t)^*\left(s_i(t)s_i(t)^*e_0\right)s_i(t)w^*v_i(t)^* = e_1q_i,$$

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etc. Since $U_i(0) = (uvS)^* p_i w p_i (uvS) w^*$, $\lim_{t\to\infty} U_i(t) = e_1 q_i$, and $[U_i(t), ze_1 q_i] \approx 0$, we have a continuous path in $\mathcal{U}(e_1 q_i (A \cap B'_{m-3}) q_i e_1)$, almost commuting with $ze_1 q_i$, from $(uvS)^* p_i w p_i (uvS) w^*$ to $e_1 q_i$. This implies the assertion for the unitary $y_i x y_i^* + e_1(1 - y_i y_i^*) = x y_i y_i^* + e_1(1 - y_i y_i^*)$. By combining these we have completed the proof.

Thus we have shown that $[x] = 0$ and $B(x, ze_1) = 0$ in the K theory of $e_1 A e_1 \cap (e_1 B_{m-3} e_1)'$. Since Ad $v\tilde{\lambda}(ze_1) \approx ze_1$ and Ad $v\tilde{\lambda}(x) \in e_1 A e_1 \cap (e_1 B_{m-4} e_1)'$, we have that $[\mathrm{Ad}\ v\tilde{\lambda}(x)] = 0$ and $B(\mathrm{Ad}\ v\tilde{\lambda}(x), ze_1) = 0$ in the K theory of $e_1 A e_1 \cap (e_1 B_{m-4} e_1)'$. Since $x_2 = x\,\mathrm{Ad}\ v\tilde{\lambda}(x)$, this conclude the proof for x_2. In this way we can conclude the proof.

Remark 2.9 Theorem 2.1 could hold for a wide class of C^*-algebras, *e.g.*, this is certainly true for a simple AT C^*-algebra of real rank zero (which is obtained as the inductive limit of finite direct sums of matrix algebras over $C^*(z)$ with z a unitary. (The proof of this fact would be simpler than of 2.1 with some modification for the choice of f_{ij}^k, v_k in the beginning of the proof of Theorem 2.1. Any two unitaries in such a C^*-algebra with the same non-trivial class in K_1 are approximately unitarily equivalent [5].)

3 Unitaries

The following result is a generalization of Proposition 2.1 of [12], where the spectrum of $u(t)$ is assumed to be finite.

Proposition 3.1 *Let A be a unital separable nuclear purely infinite simple C^*-algebra satisfying the Universal Coefficient Theorem.*

For any finite subset $\mathcal{F}$ of A and $\epsilon > 0$, there exists a finite subset $\mathcal{G}$ of A and $\delta > 0$ satisfying: For any $u \in \mathcal{U}(C[0,1] \otimes A)$ such that $\mathrm{Spec}(u(t))$ is independent of t and $\|[u(t), x]\| < \delta$ for $x \in \mathcal{G}$ and $t \in [0,1]$, there is a $v \in \mathcal{U}(C[0,1] \otimes A)$ such that $v(0) = 1$, $\|[\mathrm{Ad}\ v(t)(u(0)) - u(t)\| < \epsilon$, and $\|[v(t), x]\| < \epsilon$, $x \in \mathcal{F}$.

If $\delta > 0$ and if two subsets A and B of $\mathbf{T}$ satisfy that for any $a \in A$ there is a $b \in B$ with $|a - b| < \delta$, then we say that A is δ-contained in B. If A is δ-contained in B and B is also δ-contained in A, we say that A and B are δ-equal and write $A \overset{\delta}{\approx} B$.

Lemma 3.2 *For any $\epsilon > 0$ there is a $\delta > 0$ satisfying: If $z \in \mathcal{U}(C[0,1] \otimes A)$ satisfies that $\mathrm{Spec}(z(t)) \overset{\delta}{\approx} \mathrm{Spec}(z(0))$ for any t, then there is a $\zeta \in \mathcal{U}(C[0,1] \otimes A)$ such that $\zeta(0) = 1$ and $\|\mathrm{Ad}\ \zeta(t)(z(0)) - z(t)\| < \epsilon$, $t \in [0,1]$.*

Proof If $\mathrm{Spec}(z(t)) = \mathbf{T}$, then this is 2.4 of [12]. If $\mathrm{Spec}(z(t)) \neq \mathbf{T}$, this will follow from, *e.g.*, 2.5 of [12]. ∎

Lemma 3.3 *The above proposition is valid for a corner of a Cuntz algebra.*

Proof We will repeat the proof of Lemma 2.4 up to a certain point.

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We may assume that A is given as $e(B \times_\alpha \mathbf{Z})e$, where B is a stable AF C^*-algebra with $K_0(B) \subset \mathbf{R}$, e is a projection in B, and α is a trace-scaling automorphism of B: $\tau\alpha = \lambda\tau$ with $0 < \lambda < 1$, where τ is the trace on B defined by $\tau(p) = [p]$ for any projection $p \in \mathcal{P}(B)$ (see [15]). We may further assume that there is an increasing sequence (B_n) of finite-dimensional C^*-subalgebras of B such that $B = \overline{\cup_n B_n}$, $\alpha(B_n) \subset B_{n+1}$, $B_n \subset \alpha(B_{n+1})$, $e \in B_1$, $\alpha(e) \in B_1$, $\alpha(e) \le e$, and $\alpha(e)$ has central support e in eB_1e. Note that α has the Rohlin property and is unique up to cocycle-conjugacy [6, 3].

Let U denote the canonical unitary in $M(B \times_\alpha \mathbf{Z})$ implementing α and let $S = Ue \in A = e(B \times_\alpha \mathbf{Z})e$. Then S is an isometry in A and generates A together with eBe. We define an endomorphism λ of A by $\lambda(x) = SxS^*$, $x \in A$, whose range is $\alpha(e)A\alpha(e)$. By using the fact that the multiplication by $\alpha(e)$ on $A \cap (eB_1e)'$ is an isomorphism and the inclusion $B_n \subset \alpha(B_{n+1})$, we define a unital endomorphism $\tilde\lambda_n$ of $A \cap B'_{n+1}$ into $A \cap B'_n$ by $\tilde\lambda_n(x)\alpha(e) = \lambda(x)$ for any $n = 1, 2, \ldots$, where the notation $A \cap B'_n$ is used for $A \cap (eB_ne)'$. Since $\alpha(B_{n+1}) \subset B_{n+2}$, the range of $\tilde\lambda_n$ includes $A \cap B'_{n+2}$. We will simply denote $\tilde\lambda_n$ by $\tilde\lambda$ because $\tilde\lambda_{n+1}|A \cap B'_{n+1} = \tilde\lambda_n$.

In this situation we may specify $N, \epsilon > 0$, in place of $\mathcal{F}, \epsilon$ in the statement of the lemma, in the sense that $v \in \mathcal{U}(C[0,1] \otimes A)$ should be chosen from $C[0,1] \otimes (A \cap B'_N)$ and should satisfy $\|\tilde\lambda(v(t)) - v(t)\| < \epsilon$, $t \in [0,1]$.

Suppose that we fix N as above and $n \in \mathbf{N}$ such that $3\pi/n < \epsilon$. By the Rohlin property of α we have a *Rohlin* partition $\{e_{10}, e_{11}, \ldots, e_{1,n-1}; e_{20}, \ldots, e_{2,n}\}$ of e with $e_{\sigma,i} \in \mathcal{P}(e(B_M \cap B'_N)e)$ for some $M > N$ such that

$$\sum_{\sigma=1,2} \sum_i e_{\sigma,i} = e, \quad \max_{\sigma,i} \|\tilde\lambda(e_{\sigma,i}) - e_{\sigma,i+1}\| \approx 0.$$

(We will not be very specific about the estimates; if something is ≈ 0, then this should be appropriately close to zero.)

Note that we have fixed N, n, M as above. Let $\{E_i\}$ be the set of minimal central projections in $eB_{M+2n+2}e$ and let T_i be an isometry in A such that $T_iT_i^* \le E_i$. Let $u \in \mathcal{U}(C[0,1] \otimes A \cap B'_{M+2n+2})$ be such that $\|\tilde\lambda(u(t)) - u(t)\| \approx 0$ and $\|[u(t), T_i]\| \approx 0$. Thus $\mathcal{G}$ is the union of a family of matrix units for $eB_{M+2n+2}e$ and $\{S\} \cup \{T_i\}$ with a suitable choice of $\delta > 0$.

The last condition implies that $\mathrm{Spec}(u(t)E_i)$ is almost independent of t. Hence, by the previous lemma, there is a $v \in \mathcal{U}(C[0,1] \otimes A \cap B'_{M+2n+2})$ such that $v(0) = 1$ and $\mathrm{Ad}\, v(t)(u(0)) \approx u(t)$. Let $w(t) = v(t)^*\tilde\lambda(v(t)) \in \mathcal{U}(A \cap B'_{M+2n+1})$. Then $[w(t), u(0)] \approx 0$ and $w(0) = 1$. Let $(w_s)_{s \in [0,1]}$ denote the path in $\mathcal{U}(C[0,1] \otimes A \cap B'_{M+2n+1})$ defined by $w_s(t) = w(st)$ and note that $[w_s, 1 \otimes u(0)] \approx 0$. Let $w_0 = 1$ and $w_1 = w$ and let $w_k = w\tilde\lambda(w_{k-1})$ for $k = 2, 3, \ldots, n+1$. We can construct a rectifiable path of length at most 6π in the unitary group of

$$\{x \in C[0,1] \otimes A \cap B'_{M+n+1} \mid x(0) = 1\}$$

from w_k to 1 by using (w_s) for $k = n, n+1$ (see [14, 12]). In particular $u(0)$ almost commutes with the unitaries along the paths. By using these paths applied with $\tilde\lambda^{-k}$ for $k = 0, 1, \ldots, n$ and the Rohlin partition in $e(B_M \cap B'_N)e$, we get a $y \in \mathcal{U}(C[0,1] \otimes$

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$A \cap B_N')$ such that $w \approx y\tilde{\lambda}(y^*)$, $y(0) = 1$, and $[y, 1 \otimes u(0)] \approx 0$ (see the proof of 2.4). Then $vy \in \mathcal{U}(C[0,1] \otimes (A \cap B_N'))$ satisfies that $v(0)y(0) = 1$, $\mathrm{Ad}(v(t)y(t))(u(0)) \approx u(t)$, and $\tilde{\lambda}(v(t)y(t)) \approx v(t)y(t)$. This completes the proof. ■

Lemma 3.4 *Let z be a unitary in A with* $\mathrm{Spec}(z) = \mathbf{T}$ *and $m \in \mathbf{N}$. Then for any $\epsilon > 0$ there is a unital C^*-subalgebra $D = D_1 \oplus D_2$ of A such that $D_1 \cong M_m$, $D_2 \cong M_{m+1}$, $\|(\mathrm{Ad}\ z - \mathrm{Ad}\ U_\sigma)|D_\sigma\| < \epsilon$, where U_1 (resp, U_2) is a diagonal unitary with the eigenvalues $\{\omega \in \mathbf{C} \mid \omega^m = 1\}$ (resp. $\{\omega \in \mathbf{C} \mid \omega^{m+1} = 1\}$).*

Proof Let $e, f \in \mathcal{P}(A)$ be such that $e \neq 0$, $f \neq 0$, and $[1] = m[e] + (m+1)[f]$ and let $v \in \mathcal{U}(eAe)$ and $w \in \mathcal{U}(fAf)$ be such that $[z] = m[v] + (m+1)[w]$ and $\mathrm{Spec}(v) = \mathrm{Spec}(w) = \mathbf{T}$. We then find a family $\{s_i, t_j\}$ of partial isometries such that $s_k^*s_k = e$ for $k = 1, \dots, m$ and $t_\ell^*t_\ell = f$ for $\ell = 1, \dots, m+1$, $\sum_k s_is_i^* + \sum_\ell t_jt_j^* = 1$, and $z \approx \sum_k s_k e^{2\pi ik/m} v s_k^* + \sum_\ell t_\ell e^{2\pi\ell/(m+1)} w t_\ell^*$ (see [5, 12]). Then we define D to be the C^*-subalgebra generated by $s_is_j^*$ and $t_it_j^*$, which is a unital C^*-subalgebra isomorphic to $M_m \oplus M_{m+1}$. Since $zs_ks_\ell^* \approx e^{2\pi ik/m} s_k v s_\ell^*$ and $s_\ell^* z^* \approx e^{-2\pi i\ell/m} v^* s_\ell^*$, we have that $\mathrm{Ad}\ z(s_ks_\ell^*) \approx e^{2\pi(k-\ell)/m} s_k s_\ell$. In the same way we have that $\mathrm{Ad}\ z(t_kt_\ell^*) \approx e^{2\pi i(k-\ell)/(m+1)} t_kt_\ell^*$. Since the approximation can be made arbitrarily precise, this completes the proof. ■

Proof of Proposition 3.1 By the classification result by Kirchberg and Phillips [8, 9] there is an increasing sequence (A_n) of unital C^*-subalgebras of A with dense union such that $A_n = \bigoplus_{k=1}^{K_n} A_{nk}$ and $A_{nk} = D_{nk} \otimes C^*(z_{nk})$, where D_{nk} is of the form $e(B \times_\alpha \mathbf{Z})e$ as in the proof of 4.9 and $C^*(z_{nk})$ is the universal C^*-algebra generated by a single unitary z_{nk}. We may suppose that each $C^*(z_{nk})$ is mapped into each $A_{n+1,\ell}$ isomorphically (see [2]).

Let $\mathcal{F}$ be a finite subset of A and $\epsilon > 0$. We may suppose that $\mathcal{F}$ equals

$$\bigcup_{k=1}^{K_n} (\mathcal{F}_{nk} \cup \{z_{nk}\})$$

for some n, where $\mathcal{F}_{nk} \subset D_{nk}$. We choose $\mathcal{G}_{nk}(\subset D_{nk})$ and $\delta_{nk} > 0$ for $(\mathcal{F}_{nk}, \epsilon)$ as in Lemma 3.3. In particular $\mathcal{G}_{nk}$ contains a family of matrix units for some finite-dimensional C^*-subalgebra B_{nk}.

Let E_{nk} denote the identity of A_{nk}. We choose a unital C^*-subalgebra $D_k = D_{k1} \oplus D_{k2}$ (with $D_{k1} \cong M_m$ and $D_{k2} \cong M_{m+1}$) of $E_{nk}AE_{nk}$ for z_{nk}, for a large m as in the previous lemma. We may suppose that D_k commutes with the above B_{nk}. Let C_{nk} denote the set of matrix units of D_k and let T_k be an isometry in A such that $T_kT_k^* \leq E_{nk}$. Let also T_{ki} be an isometry in $E_{nk}A \cap B_{nk}'E_{nk}$ for $i = 1, 2$ such that $T_{k1}T_{k1}^* \leq 1_{D_{k1}}$ and $T_{k2}T_{k2}^* \leq 1_{D_{k2}}$. We set

$$\mathcal{G} = \bigcup_{k=1}^{K_n} \left(\mathcal{G}_{nk} \cup C_{nk} \cup \{z_{nk}, T_k, T_{k1}, T_{k2}\} \right).$$

We will take a sufficiently small $\delta > 0$.

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Let $u \in \mathcal{U}(C[0,1] \otimes A)$ be such that $\|[u(t), x]\| < \delta$, $x \in \mathcal{G}$ and $\mathrm{Spec}(u(t))$ is independent of t. Since $u(t)$ almost commutes with E_{nk} and T_k, we may suppose that $[u(t), E_{nk}] = 0$ and that $\mathrm{Spec}(u(t)E_{nk})$ is almost independent of t and discuss each $uE_{nk} \in \mathcal{U}(C[0,1] \otimes E_{nk}AE_{nk})$ separately. Denoting $E_{nk}AE_{nk}$ by A, we have reached the following situation:

$$e(B \times_\alpha \mathbf{Z})e \subset A, \quad B = \overline{\cup_m B_m}, \quad u(t) \in A \cap (eB_{M+2n+2}e)' \cap D',$$

$$\tilde{\lambda}(u(t)) \approx u(t), \quad \mathrm{Spec}(u(t)f) = \mathrm{Spec}(u(0)f), \quad [u(t), z] \approx 0,$$

for each minimal central projection f in $eB_{M+2n+2}e \vee D$, where $e \in B$ is the identity of A and D ($\cong M_m \oplus M_{m+1}$) denotes the unital finite-dimensional C^*-subalgebra of $A \cap (eB_{M+2n+2}e)'$ associated with z.

We then find a $v \in \mathcal{U}(C[0,1] \otimes (eB_{M+2n+2}e)' \cap D')$ such that $v(0) = 1$ and $\mathrm{Ad}\, v(t)(u(0)) \approx u(t)$. If $w(t) = v(t)^* z v(t) z^*$, it follows that $w(0) = 1$ and $[w(t), u(0)] \approx 0$. By using the Rohlin property for $\mathrm{Ad}\, z|D$, we obtain a $y \in C[0,1] \otimes A \cap (eB_{M+2n+2}e)'$ such that $w = v^* z v z^* \approx yzy^* z^*$, $y(0) = 1$, and $[y(t), u(0)] \approx 0$. Then $\mathrm{Ad}\, z(vy) \approx vy$, $v(0)y(0) = 1$, and $\mathrm{Ad}(v(t)y(t))(u(0)) \approx u(t)$. We shall denote vy by v.

We thus have $v \in \mathcal{U}(C[0,1] \otimes A \cap (eB_{M+2n+2}e)')$ such that $v(0) = 1$, $\mathrm{Ad}\, v(t)(u(0)) \approx u(t)$, and $[v(t), z] \approx 0$. Note that $[v(t)^* \tilde{\lambda}(v(t)), u(0)] \approx 0$. By using the Rohlin property for $\tilde{\lambda}$ we obtain a $y \in \mathcal{U}(C[0,1] \otimes A \cap (eB_N e)')$ such that $v^* \tilde{\lambda}(v) \approx y \tilde{\lambda}(y^*)$, $y(0) = 1$, and $[y(t), u(0)] \approx 0$. Then vy satisfies the desired conditions. ∎

4 Rohlin Flows

We recall the definition of the Rohlin property for flows [10], where $M(A)$ denotes the multiplier algebra of A.

Definition 4.1 Let A be a C^*-algebra and α a flow on A. The flow α is said to have the *Rohlin property* if for any $p \in \mathbf{R}$ there is a sequence (u_n) in $\mathcal{U}(M(A))$ such that $\|\alpha_t(u_n) - e^{ipt} u_n\| \to 0$ uniformly in t on every compact subset of $\mathbf{R}$ and $\|[u_n, x]\| \to 0$ for any $x \in A$.

In the following ω denotes a free ultrafilter on $\mathbf{N}$ and A^ω is the quotient of $\ell^\infty(A)$ divided by the ideal $c^\omega(A) = \{x = (x_n) \mid \lim_{n\to\omega} \|x_n\| = 0\}$. See Section 1 for details including the definition of A^ω_α when α is a flow on A. The K_0 version of the following result is shown in [12].

Lemma 4.2 *Let α be a Rohlin flow on A. Then for any unitary $u \in A' \cap A^\omega$ there is a unitary $v \in (A' \cap A^\omega_\alpha)^\alpha$ such that $[u] = [v]$ in $K_1(A' \cap A^\omega)$.*

Proof Let $u \in \mathcal{U}(A' \cap A^\omega)$ and let (u_n) be a sequence in $\mathcal{U}(A)$ which represents u. Fix a large $T > 0$. By 3.1 there is a sequence (V_n) in $\mathcal{U}(C[0,T] \otimes A)$ such that $\max_t \|\mathrm{Ad}\, V_n(t)(u_n) - \alpha_t(u_n)\|$ converges to zero as $n \to \omega$ and $\max_t \|[V_n(t), x]\| \to 0$ as $n \to \omega$ for any $x \in A$. By [14] (or 2.7 of [12]) there is a sequence (v_n) in

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$\mathcal{U}(C[0,T]\otimes A)$ such that $v_n(0)=1$, $v_n(T)=V_n(T)^*$, $(v_n)\in A'\cap(C[0,T]\otimes A)^\omega$, and the length of $(v_n(t))_{t\in[s_1,s_2]}$ is less than $6\pi|s_2-s_1|/T$ for any $0\le s_1<s_2\le T$.

We define a unitary $U_n\in C(\mathbf{R}/T\mathbf{Z})\otimes A$ by setting

$$U_n(t)=\alpha_{t-T}(v_n(t))\alpha_t(u_n)\alpha_{t-T}(v_n(t)^*)$$

for $t\in[0,T]$ except for t close to T. Since $U_n(T)\approx u_n=U_n(0)$, this indeed defines a unitary in $C(\mathbf{R}/T\mathbf{Z})\otimes A$ by suitably defining $U(t)\approx u_n$ for $t\approx T$ and it follows that $(U_n)\in A'\cap(C(\mathbf{R}/T\mathbf{Z})\otimes A)^\omega$.

Define a unitary w_n in $C(\mathbf{R}/T\mathbf{Z})\otimes A$ by $w_n(t)=\alpha_{t-T}(v_n(t))V_n(t)$, where $w_n(T)=1=w_n(0)$. Then it follows that $\|U_n-w_n(1\otimes u_n)w_n^*\|\to 0$ as $n\to\omega$.

If γ denotes the flow on $C(\mathbf{R}/T\mathbf{Z})$ defined by $(\gamma_t f)(s)=f(s-t)$, it follows, as in the proof of 3.1 of [12], that

$$\|\gamma_t\otimes\alpha_t(U_n)-U_n\|\le 12\pi|t|/T+\epsilon_n,$$

where $\epsilon_n\to 0$ as $n\to\infty$.

Let (u_j) be a central sequence in $\mathcal{U}(A)$ such that $\|\alpha_t(u_j)-e^{2\pi it/T}u_j\|\to$ uniformly in t on every compact subset. We define a linear map ϕ_j from the algebraic tensor product $C(\mathbf{R}/T\mathbf{Z})\odot A$ into A by $\phi_j(z^\ell\otimes a)=u_j^\ell a$, where z is the canonical unitary in $C(\mathbf{R}/T\mathbf{Z})$. Then (ϕ_j) is an approximate homomorphism of $C(\mathbf{R}/T\mathbf{Z})\odot A$ into A in the sense that $\|\phi_j(xy)-\phi_j(x)\phi_j(y)\|\to 0$, $\|\phi_j(x)^*-\phi_j(x^*)\|\to 0$, and $\|\phi_j(x)\|\to\|x\|$ for any $x,y\in C(\mathbf{R}/T\mathbf{Z})\odot A$. It also follows that (ϕ_j) intertwines $\gamma_t\otimes\alpha_t$ and α_t: $\|\phi_j(\gamma_t\otimes\alpha_t)(x)-\alpha_t\phi_j(x)\|\to 0$ for $x\in C(\mathbf{R}/T\mathbf{Z})\odot A$. By using these facts we can define a unitary u_n' as a kind of $\phi_j(U_n)$ for a large j. At the same time we may suppose that we can define a unitary w_n' as a kind of $\phi_j(w_n)$; we then have that $u_n'\approx \mathrm{Ad}\ w_n'(u_n)$ as $\phi_j(1\otimes u_n)=u_n$. In this way we get a sequence (u_n') in $C(\mathbf{R}/T\mathbf{Z})\otimes A$ such that $\lim_{n\to\omega}\|\alpha_t(u_n')-u_n'\|\le 12\pi|t|/T$, $\lim_{t\to\omega}\|[u_n',x]\|=0$ for $x\in A$, and $[(u_n')]=[(u_n)]$ in $K_1(A'\cap A^\omega)$. By taking a larger and larger T we can obtain the desired sequence which belongs to $\mathcal{U}((A'\cap A_\alpha^\omega)^\alpha)$. (See [12] for details.) ■

Lemma 4.3 *Let α be a Rohlin flow on A. Then for any unitary $u\in A$ there are sequences (u_n') and (v_n) in $\mathcal{U}(A)$ such that $\|\alpha_t(u_n')-u_n'\|\to 0$ uniformly in t on every compact subset of $\mathbf{R}$ and $\|v_nuv_n^*-u_n'\|\to 0$.*

Proof This follows from the proof of 4.2. ■

Lemma 4.4 *Let $u,v\in\mathcal{U}((A'\cap A_\alpha^\omega)^\alpha)$. If $[u]=[v]$ in $K_1(A'\cap A^\omega)$, then $[u]=[v]$ in $K_1((A'\cap A_\alpha^\omega)^\alpha)$.*

Proof Suppose that $[u]=0$ in $K_1(A'\cap A^\omega)$ and let (u_n) be a sequence in $\mathcal{U}(A)$ representing u. Since $A'\cap A^\omega$ is purely infinite and simple [8], we can approximate u by a unitary with finite spectrum in $A'\cap A^\omega$ [17]. Then we can argue as in 3.2 of [12] using 3.6 there. That is, we can approximate each u_n by a unitary with finite spectrum whose spectral projections are almost α-invariant. Thus each u_n is connected to 1 by a rectifiable path in $\mathcal{U}(A)$ of length about π which is almost α-invariant. In this way we can find a path in $\mathcal{U}((A'\cap A_\alpha^\omega)^\alpha)$ which connects u and 1.

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The previous paragraph is sufficient for the conclusion. But supposing that $[u] = [v] \neq 0$ in $K_1(A' \cap A^\omega)$, we shall give a detailed proof using 3.1 and [10]. Since $A' \cap A^\omega$ is a unital purely infinite simple C^*-algebra, u and v are in the same connected component in $\mathcal{U}(A' \cap A^\omega)$. Let $(U(t))_{t\in[0,1]}$ be a continuous path in $\mathcal{U}(A' \cap A^\omega)$ such that $U(0) = u$ and $U(1) = v$. Let (U_n) be a sequence in $\mathcal{U}(C[0,1] \otimes A)$ representing U. Then by 3.1 there is a sequence (V_n) in $\mathcal{U}(A)$ such that $\max_t \|V_n(t)U_n(0)V_n(t)^* - U_n(t)\| \to 0$ as $n \to \omega$ and $\max_t \|[V_n(t), x]\| \to 0$ as $n \to \omega$ for all $x \in A$. Let $z_n = V_n(1)$. Then $(z_n) \in \mathcal{U}(A' \cap A^\omega)$ and $\|z_n u_n z_n^* - v_n\| \to 0$ as $n \to \infty$. Let $w_n(t) = z_n^* \alpha_t(z_n)$. Then (w_n) is a sequence of α-cocycles such that $\|[w_n(t), x]\| \to 0$ as $n \to \omega$ uniformly in t on every compact subset and $\|[w_n(t), u_n]\| \to 0$ as $n \to \omega$. Then there is a sequence (y_n) in $\mathcal{U}(A)$ such that $(y_n) \in \mathcal{U}(A' \cap A^\omega)$, $\|[y_n, u_n]\| \to 0$ as $n \to \omega$, and $\sup_{t\in[0,1]} \|w_n(t) - y_n \alpha_t(y_n^*)\| \to 0$ as $n \to \omega$ [12, 10]. Since $(z_n y_n) \in \mathcal{U}((A' \cap A^\omega_\alpha)^\alpha)$ and $\|(z_n y_n) u_n (z_n y_n)^* - v_n\| \to 0$ as $n \to \omega$, this implies that $[u] = [v]$ in $K_1((A' \cap A^\omega_\alpha)^\alpha)$. ∎

If α is a flow, then α_t is homotopic to the identity and so often is approximately inner for each $t \in \mathbf{R}$. The following is defined in [12].

Definition 4.5 Let A be a C^*-algebra and α a flow on A. Then α_t is said to be *α-invariantly approximately inner* if there is a sequence (u_n) in $\mathcal{U}(A)$ such that $\alpha_t = \lim \operatorname{Ad} u_n$ and $\|\alpha_s(u_n) - u_n\|$ converges to zero uniformly in s on every compact subset.

Theorem 4.6 *Let A be a unital separable nuclear purely infinite simple C^*-algebra satisfying* UCT *and let α be a flow on A. Then the following conditions are equivalent.*

(1) *α has the Rohlin property.*
(2) *$(A' \cap A^\omega_\alpha)^\alpha$ is purely infinite and simple, $K_0((A' \cap A^\omega_\alpha)^\alpha) \cong K_0(A' \cap A^\omega)$ induced by the embedding, and* $\operatorname{Spec}(\alpha|A' \cap A^\omega_\alpha) = \mathbf{R}$.
(3) *The crossed product $A \times_\alpha \mathbf{R}$ is purely infinite and simple and the dual action $\hat{\alpha}$ has the Rohlin property.*
(4) *The crossed product $A \times_\alpha \mathbf{R}$ is purely infinite and simple and each α_t is α-invariantly approximately inner.*

If the above conditions are satisfied, it also follows that $K_1((A' \cap A^\omega_\alpha)^\alpha) \cong K_1(A' \cap A^\omega)$, which is induced by the embedding.

When α is a flow on A, we denote by δ_α the infinitesimal generator of α, which is a closed derivation in A. If $h \in A_{sa}$, then ad ih is a bounded derivation. We denote by $\alpha^{(h)}$ the flow generated by $\delta_\alpha + \operatorname{ad} ih$. See [4, 16] for details.

Proposition 4.7 *Let A be a non-unital separable nuclear purely infinite simple C^*-algebra satisfying the* UCT. *Then the following conditions are equivalent.*

(1) *α has the Rohlin property.*
(2) *For any $\epsilon > 0$ there exists an $h \in A_{sa}$ and an increasing sequence (e_n) in $\mathcal{P}(A)$ such that $\|h\| < \epsilon$, $\alpha^{(h)}_t(e_n) = e_n$, $\alpha^{(h)}|e_n A e_n$ has the Rohlin property, and (e_n) is an approximate identity for A.*

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Proof Suppose (2). Then it follows that $\alpha^{(h)}|(e_n - e_{n-1})A(e_n - e_{n-1})$ has the Rohlin property for all n with $e_0 = 0$. We choose, for any $p \in \mathbf{R}$, a central sequence $(u_{n,m})$ in $\mathcal{U}((e_n - e_{n-1})A(e_n - e_{n-1}))$ such that $\|\alpha_t(u_{n,m}) - e^{ipt}u_{n,m}\|$ converges to zero, as $m \to \infty$, uniformly in t on every compact subset of $\mathbf{R}$. By passing to a subsequence we may suppose that $\|\alpha_t(u_{n,m}) - e^{ipt}u_{n,m}\| < 1/m$ for $|t| \le 1$. Let $u_m = \sum_{n=1}^{\infty} u_{n,m}$, which converges in the multiplier algebra $M(A)$. Then (u_m) is the desired sequence in $\mathcal{U}(M(A))$ for $p \in \mathbf{R}$.

Suppose (1). Let $p \in \mathcal{P}(A)$ and fix a large $T > 0$. Then there exists a projection $f \in A$ such that $\alpha_{-t}(f)p \approx p$ for any $t \in [0, T]$. Again there exists a projection $e \in A$ such that $\alpha_t(e)f \approx f$ for any $t \in [0, T]$. Let f_t be the support projection of $\alpha_t(e)f\alpha_t(e)$. Then $t \mapsto f_t$ is continuous and $f_t \le \alpha_t(e)$ and $f_t \approx f$ for $t \in [0, T]$. Let u_t denote the unitary part of the polar decomposition of $f_t f_0 + (1 - f_t)(1 - f_0)$; then $u_t \approx 1$ and $\mathrm{Ad}\, u_t^*(f_t) = f_0$ for $t \in [0, T]$. We find a continuous function $t \mapsto v_t \in \mathcal{U}(A)$ such that $\mathrm{Ad}\, v_t(e - f_0) = \mathrm{Ad}\, u_t^*(\alpha_t(e)) - f_0$ and $v_t f_0 = f_0$. Let $w_t = u_t v_t$. Then $w_t f \approx f$ and $\mathrm{Ad}\, w_t(e) = \alpha_t(e)$ for $t \in [0, T]$.

We find a rectifiable path $(y_t)_{t\in[0,T]}$ in $\mathcal{U}(A)$ such that $y_0 = 1$, $y_T = w_T^*$, $y_t f \approx f$, and the length of $(y_t)_{t\in[s_1,s_2]}$ is dominated by $6\pi(s_2 - s_1)/T$, because we can construct such a path in terms of (w_t) (see [14, 12]). We then define a projection E in $C(\mathbf{R}/T\mathbf{Z}) \otimes A$ by

$$E(t) = \alpha_{t-T}(y_t)\alpha_t(e)\alpha_{t-T}(y_t)^*,$$

which satisfies that $E(0) = e = E(T)$. Since $p\alpha_{t-T}(y_t) \approx p\alpha_{t-T}(fy_t) \approx p$, we obtain that $E(t)p \approx p$. By using the Rohlin property for α we have an approximate homomorphism (ϕ_j) of $C(\mathbf{R}/T\mathbf{Z}) \odot A$ into A such that $\alpha_t\phi_j \approx \phi_j(\gamma_t \otimes \alpha_t)$, where γ is the flow on $C(\mathbf{R}/T\mathbf{Z})$ induced by translations (see the proof of 4.2). Applying ϕ_j to E, we get a projection e' in A such that $\|\alpha_t(e') - e'\| < 6\pi/T + \epsilon$ for $t \in [0, 1]$ and $e'p \approx p$. By perturbing e' slightly we may assume that $\|\delta_\alpha(e')\|$ is small (depending on $1/T$) (see [4, 16]). In this way we can construct an approximate identity (e_n) consisting projections such that $\|\delta_\alpha(e_n)\| \to 0$ and $\|e_n p - p\| \to 0$. It is then easy to show the conclusion. ■

Proof of Theorem 4.6

The last statement follows from 4.2 and 4.4.

We have shown that (1)⇔(2)⇒(4) in [12].

It is easy to show that (4) implies (3). Let $t \in \mathbf{R}$ and let (u_n) be a sequence in $\mathcal{U}(A)$ such that $\alpha_t = \lim \mathrm{Ad}\, u_n$ and $\|\alpha_s(u_n) - u_s\| \to 0$ uniformly in s on every compact subset of $\mathbf{R}$. If we denote by $\lambda(\,\cdot\,)$ the canonical unitary flow in $M(A \times_\alpha \mathbf{R})$ implementing α, then we have that $\hat{\alpha}_p(u_n^*\lambda(t)) = e^{ipt}u_n^*\lambda(t)$ and $\|[u_n^*\lambda(t), x]\| \to 0$ for any $x \in A \times_\alpha \mathbf{R}$.

Suppose (3). By the previous proposition we have an $h = h^* \in A \times_\alpha \mathbf{R}$ and an increasing sequence (e_n) in $\mathcal{P}(A \otimes_\alpha \mathbf{R})$ such that (e_n) is an approximate identity and $\hat{\alpha}_p^{(h)}(e_n) = (e_n)$ and $\beta = \hat{\alpha}^{(h)}|e_n(A \times_\alpha \mathbf{R})e_n$ has the Rohlin property. Then from (1)⇒(3), we obtain that the dual flow of β has the Rohlin property. Since $e_n(A \times_\alpha \mathbf{R})e_n \times_\beta \mathbf{R} = e_n(A \times_\alpha \mathbf{R} \times_{\hat{\alpha}} \mathbf{R})e_n$ with the dual flow $\hat{\beta}$ being a restriction of $\hat{\hat{\alpha}}$ and (e_n) is a sequence in $M(A \times_\alpha \mathbf{R} \times_{\hat{\alpha}} \mathbf{R})$, we can conclude that $\hat{\hat{\alpha}}$ has the Rohlin property. By the Takesaki-Takai duality we have that $A \times_\alpha \mathbf{R} \times_{\hat{\alpha}} \mathbf{R} \cong A \otimes K(L^2(\mathbf{R}))$

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and $\hat{\alpha}_t = \alpha_t \otimes \mathrm{Ad}\ \lambda(-t)$, where $K(L^2(\mathbf{R}))$ denotes the compact operators on $L^2(\mathbf{R})$. Then it follows that α has the Rohlin property. ∎

Let α and β be flows on a unital C^*-algebra A. We say that α is an approximate cocycle perturbation of β if there is a sequence (u_n) of β-cocycles such that

$$\alpha_t(x) = \lim_{n\to\infty} \mathrm{Ad}\ u_n(t)\beta_t(x)$$

uniformly in t on every compact subset of $\mathbf{R}$ for any $x \in A$ [11]. If α is an approximate cocycle perturbation of the trivial flow id, then α is approximately inner, *i.e.,* $\alpha_t = \lim \mathrm{Ad}\ e^{ith_n}$ for some sequence (h_n) in A_{sa}. A Rohlin flow is never approximately inner. The following result generalizes 4.4 of [11].

Proposition 4.8 *Let A be a unital separable nuclear purely infinite simple C^*-algebra satisfying the Universal Coefficient Theorem and let α be a Rohlin flow on A. Then the trivial flow* id *is an approximate cocycle perturbation of α. In particular there is a unital approximately inner endomorphism ϕ of A such that $\phi = \mathrm{Ad}\ u_t\alpha_t\phi$ for some α-cocycle u.*

Lemma 4.9 *Let D be a finite-dimensional C^*-subalgebra of A. Then there is a α-cocycle u such that $\mathrm{Ad}\ u_t\alpha_t(x) = x$ for any $x \in D$.*

Proof See [4, 16] for example. We do not need the Rohlin property for this. ∎

Lemma 4.10 *Let z be a unitary. Then for any $\epsilon > 0$ there is an α-cocycle u such that $\|\mathrm{Ad}\ u_t\alpha_t(z) - z\| < \epsilon$ for $t \in [0,1]$.*

Proof By 4.3 for any $\epsilon > 0$ there are $Z, v \in \mathcal{U}(A)$ such that $\|\alpha_t(Z) - Z\| < \epsilon$ for $t \in [0,1]$ and $\|vzv^* - Z\| < \epsilon$. Let $u_t = v^*\alpha_t(v)$, which is an α-cocycle. Then it follows that $\|\mathrm{Ad}\ u_t\alpha_t(z) - z\| < 3\epsilon$ for $t \in [0,1]$. ∎

Proof of Proposition 4.8 The last statement follows from 4.6 of [11].

We may suppose that there is an increasing sequence (A_n) of C^*-subalgebras of A with dense union such that each A_n is a finite direct sum of C^*-algebras of the form $\mathcal{O} \otimes C^*(z)$, where $\mathcal{O}$ is a corner of a Cuntz algebra and $C^*(z)$ is the C^*-algebra generated by a unitary with full spectrum. We assume that $\mathcal{O}$ is given as $e(B \times_\gamma \mathbf{Z})e$, where B is a stable AF C^*-algebra with $K_0(A) \subset \mathbf{R}$, γ is a trace-scaling automorphism of B, and $e \in \mathcal{P}(B)$, as in the proof of 2.1.

It suffices to show that there is a sequence (u_n) of α-cocycles such that $\|\mathrm{Ad}\ u_n(t)\alpha_t(x) - x\| \to 0$ uniformly in $t \in [0,1]$ for any $x \in A_1$. It again suffices to show this assuming that $A_1 = e(B \times_\gamma \mathbf{Z})e \otimes C^*(z)$.

Suppose that B is the completion of the union of an increasing sequence (B_n) of finite-dimensional C^*-algebras such that $e, \gamma(e) \in B_1$, $\gamma(e) \le e$, and the central support of $\gamma(e)$ in eB_1e is e. Moreover we assume that $\gamma^{\pm}(B_n) \subset B_{n+1}$. We denote by U the canonical unitary in $M(B \times_\gamma \mathbf{Z})$ implementing γ and set $S = Ue$, which is an isometry in $e(B \times_\gamma \mathbf{Z})e$. By Lemmas 4.9 and 4.10 we may assume, for a large n and

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a sufficiently small $\epsilon > 0$, that $\alpha_t|B_{n+1} = \mathrm{id}$ and $\|\alpha_t(z) - z\| < \epsilon$ for $t \in [0,1]$. We shall show that there is an α-cocycle u in $A \cap B_1'$ such that $\| \mathrm{Ad}\ u_t\alpha_t(S) - S\| \approx 0$ and $\|[u_t, z]\| \approx 0$ for $t \in [0,1]$.

Let $w_t = S^*\alpha_t(S)$. Since $\alpha_t(SS^*) = SS^* \in B_1$, (w_t) is an α-cocycle. If $x \in B_n$, then $xw_t = xS^*\alpha_t(S) = S^*\lambda(x)\alpha_t(S) = S^*\alpha_t(\lambda(x)S) = S\alpha_t(S)x$, where $\lambda(x) = SxS^* \in B_{n+1}$. Thus $w_t \in A \cap B_n'$. We also have that $\|[w_t, z]\| < 2\epsilon$ for $t \in [0,1]$. Then we find a $v \in \mathcal{U}(A \cap B_n')$ such that $\|w_t - v\alpha(v^*)\| \approx 0$ and $\|[v, z]\| \approx 0$ (but in general is much bigger than ϵ). Then it follows that $\alpha_t(Sv) \approx Sv$ for $t \in [0,1]$.

The above v is obtained as follows [10]. Take a large T such that both $1/T$ and $T\epsilon$ are small and define a unitary $V \in C(\mathbf{R}/T\mathbf{Z}) \otimes A$ by

$$V(t) = w_t\alpha_{t-T}(x(t)^*),$$

where $(x(t))_{t\in[0,T]}$ is a path in $\mathcal{U}(A)$ such that $x(0) = 1$, $x(T) = w_T$, and $\|x(s) - x(t)\| < 6\pi|s - t|$ for $s, t \in [0, T]$. Since such a path is obtained in terms of w_t and sufficiently central elements in A, we may suppose that $x(t) \in A \cap B_n'$ and $[x(t), z] \approx 0$ (of the order ϵT). Moreover it follows that $[V] = 0$ in $K_1(C(\mathbf{R}/T\mathbf{Z})\otimes A)$. (We can see this by making T decrease to zero; the construction of $(x(t))_{t\in[0,T]}$ from $(w_t)_{t\in[0,T]}$ is canonical.) Then we get v as an image of an approximate homomorphism of $C(\mathbf{R}/T\mathbf{Z})\otimes A$ into A as in the proof of 4.2. Since the Bott element $B(V, 1\otimes z)$ is zero in $K_0(A \cap B_n')$, which follows from $V(0) = 1$, the same follows for the pair v and z in $A \cap B_n'$. It also follows that $[v] = 0$ in $K_1(A \cap B_n')$.

By using the above facts and the Rohlin property for $\tilde{\lambda}$ as in the proof of 2.1, we find a $y \in \mathcal{U}(A \cap B_1')$ such that $\tilde{\lambda}(v) \approx y\tilde{\lambda}(y^*)$ and $[y, z] \approx 0$. We define $u_t = y^*\alpha_t(y)$. Since $Sv \approx ySy^*$, we have that $\mathrm{Ad}\ u_t\alpha_t(S) \approx S$ and $[u_t, z] \approx 0$ for $t \in [0,1]$. This concludes the proof.

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