P 121. We will say of two sets A, B in a topological space, that B is "peripheral" to A, if:

(a) the closure of A contains B, and

(b) the closure of B has no points in common with A.

It is easily seen that this relation is transitive. Find, in a Hausdorff space, a collection of sets which is linearly ordered by "peripheral" and has the order-type of the reals.

M. Shimrat, York University

Solution by C. J. Knight, The University of Sheffield

The problem posed asks for a collection of subsets of a Hausdorff space, ordered by 'B is peripheral to A' in the order type of the reals. In fact, such a collection can be exhibited in any given partial ordering.

We recall that B is said to be peripheral to A if $B \subseteq \bar{A}$ and $\bar{B} \cap A = \phi$. This relation is clearly transitive, and, provided we exclude the empty set, irreflexive.

THEOREM Let $<$ be a transitive irreflexive relation on the set E. Then there exists a compact Hausdorff space X and a family $\{A_t | t \in E\}$ of distinct subsets of X, such that A_t is peripheral to A_r if and only if $t < r$.

Proof: Let X be the Cartesian product of copies of $[0,1]$, one for each member e of E. We write the members of X as functions, so that if $x \in X$ then $x(e) \in [0,1]$ for each e in E. If a and b are members of E, we write $a \leqslant b$ when either $a < b$ or $a = b$.

For each t in E, let $A_t = \{x \in X | x(e) < 1$ if $e \leqslant t$, and $x(e) = 1$ otherwise$\}$.

Distinct elements t give distinct sets A_t. For, if $t \neq t_1$ then one at least of $t \leqslant t_1$ and $t_1 \leqslant t$ is false.

Suppose that $t \leqslant t_1$ is false, and let $z(e) = {}^1/2$ if $e \leqslant t_1$ and $z(e) = 1$ otherwise; then $z \in A_{t_1} \setminus A_t$.

Clearly,

$$\bar{A}_t = \{x \in X \mid x(e) = 1 \text{ whenever it is false that } e \leqslant t\} .$$

Suppose now that $t < r$, that $x \in A_t$ and that it is false that $e \leqslant r$. Then it is certainly false that $e \leqslant t$, so we have $x(e) = 1$, and so $x \in \bar{A}_r$.
Thus $A_t \subseteq \bar{A}_r$. Moreover, if $y \in \bar{A}_t$, then $y(r) = 1$, and if $y \in A_r$ then $y(r) < 1$, and thus $\bar{A}_t \cap A_r = \phi$. So we have proved that if $t < r$ then A_t is peripheral to A_r.

Suppose, on the other hand, that A_t is peripheral to A_r, and let $w(e) = {}^1/2$ whenever $e \leqslant t$, and $w(e) = 1$ otherwise. Then $w \in A_t$, and hence $w \in \bar{A}_r$. However, since $w(t) \neq 1$, this implies that $t \leqslant r$. But t and r cannot be equal, since then A_t could not be peripheral to A_r. Thus $t < r$. This completes the proof of the theorem.

The space X constructed is not metrizable unless E is countable. This fact suggests the following problem:

Which partial order-types can be realised by the relation B' is peripheral to A' between some of the subsets of a <u>metric</u> space? In particular, is the order-type of R realisable in this way?

Also solved by A.C. Thompson, J. Washenberger, and the proposer.

<u>P 122</u>. Suppose G is a topological group, K a compact set and V a neighbourhood of the identity in G. Is there a positive integer N depending on K and V such that K contains no more than N non overlapping translates of V?

J.B. Wilker, University of Toronto

https://doi.org/10.1017/S0008439500028459 Published online by Cambridge University Press

Solution by M. Edelstein, Dalhousie University

Let U be an open neighbourhood of the identity with $U^{-1} = U$ and $UU \subset V$. If $y \in G$ and $x, z \in yU$ then $z \in xV$. Hence if $\{y_i U: i = 1, 2, \ldots, M\}$ covers K each translate xV, $x \in K$, must contain some $y_i U$. Thus we can take $N = M$.

Also solved by H.B. Secord, J.E. Marsden, R. Iltis and the proposer.

P 123. Let $u^1, u^2, \ldots$ be sequences, $u^i = \{u^i_n\}$, such that, for each i, $\Sigma_n |u^i_n|^p < \infty$ if and only if $p > 1$.

(Example: $\{\frac{1}{n}\}$, $\{\frac{1}{n \log n}\}$, $\{\frac{1}{n \log n \log \log n}\}$; ...).

Show that there exists a sequence x such that $\Sigma_n x_n u^i_n$ is convergent for each i, and $x_n \to 0$, but $\Sigma |x_n|^p = \infty$ for all $p \geq 1$.

A. Wilansky, Lehigh University

Solution by B.L.D. Thorp, York University

For each positive integer m let

$$S_m = \{y = \{y_n\} : \Sigma_n u^m_n y_n \text{ converges}\} .$$

Then, for each m, S_m is an FK space [1; Lemma 1, p.227] and $\ell^p \subset S_m$ ($p = 1, 2, \ldots$). The set $S = c_0 \cap \bigcap_{m=1}^{\infty} S_m$ is an FK space [1; Theorem 3, p.205] and $\bigcup_{n \in w} \ell^n \subset S$. Since $\bigcup_{n \in w} \ell^n$ is not an FK space [1; Cor. 6, p.205], $S \setminus \bigcup_{n \in w} \ell^n$

is non-empty and any element of this set satisfies the required condition.

REFERENCE

1. A. Wilansky, Functional Analysis, Blaisdell (1964).

Also solved by the proposer.

https://doi.org/10.1017/S0008439500028459 Published online by Cambridge University Press