Can. J. Math., Vol. XXVIII, No. 5, 1976, pp. 929–937

MORE ABOUT THE MATHIEU GROUP M_{22}

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1. Introduction. We assume familiarity with the notation and contents of Conway [**2**] and Edge [**3**]. That the Mathieu group is a subgroup of the simple group PSU $(6, 2^2)$ appears to have been first recognized by Conway and is consequent upon his identification of $\cdot 222$ with PSU $(6, 2^2)$. Although we know of no proof of this identification in the literature, several proofs exist in the folklore of the subject: for example, N. Patterson showed one of the authors a proof that depends on $\cdot 222$ being a Fischer group, hence on consideration of order, isomorphic to PSU $(6, 2^2)$. There is another proof which relies on McLaughlin's work on rank three groups.

The group $C_3 \cdot M_{22} = M_{22}{}^*$ is contained in SU $(6, 2^2)$ and this group is the subject of the geometric considerations which lead to unitary generators for $M_{22}{}^*$. The Schur multiplier of M_{22} has order six [**1**].

An involutory outer automorphism of $M_{22}{}^*$, which is induced by that of M_{22} and commutes with the centre, is displayed.

2. The unitary geometry and the Steiner system. In the Leech lattice Λ, as described in [**2**] and [**3**], $L = (0^{24})$, $M = (-3, 1, 1^{22})$ and $N = (1, -3, 1^{22})$ are the vertices of a triangle of type 222, the edges having squared length $32 = 16 \cdot 2$. Recall that a vector in Λ is of type n if it has squared length $16n$. The Leech lattice is acted on by M_{24} through permutation of the 24 coordinates. Thus the stabilizer of the first two coordinates contains M_{22} but also fixes the triangle LMN, hence $M_{22} \subseteq \cdot 222$, and the outer automorphism of M_{22} is induced by an automorphism of $\cdot 222 = PSU$ $(6, 2^2)$ resulting from the interchange of two of the vertices of the triangle.

Edge [**3**] gathers evidence for a geometric proof of the isomorphism $\cdot 222 = PSU$ $(6, 2^2)$ and shows that it is possible to find geometric objects in the Leech lattice thus proving that $\cdot 222$ has transitive permutation representations on 693, and 6237 as well as 891 letters. These correspond to the permutation representations of PSU $(6, 2^2)$ on the 693 isotropic points, 6237 totally isotropic lines and the 891 totally isotropic planes respectively. It is the representation of $\cdot 222$ on 891 letters that is easiest to describe which gives the clue leading to the description of the Steiner system S $(3, 6, 22)$ inside projective unitary space of dimension five (as usual five-dimensional projective geometry is described as the lattice of non-zero subspaces of a six-dimensional vector space) over the field F_4.

Received June 10, 1975 and in revised form, February 10, 1976. This research was supported by the National Research Council of Canada.

Let LMN be a triangle of type 222 and $\mathscr{T}$ the set of points T_i of the Leech lattice such that $LMNT_i$ is a tetrahedron of type 222222. Edge shows that $\mathscr{T}$ contains 891 points. Let $\pi_i \rightarrow T_i$ be a putative correspondence between totally isotropic planes and members of $\mathscr{T}$. It is possible to interpret statements about intersections of planes $\pi_i \wedge \pi_j$ as statements about the length of vectors T_iT_j thus:

(2.11) $\pi_1 \wedge \pi_2$ is a point————T_1T_2 is of type 2,

(2.12) $\pi_1 \wedge \pi_2$ is a line————T_1T_2 is of type 4,

(2.13) π_1 is skew to π_2————T_1T_2 is of type 3.

Edge [**3**] shows that this can be justified in so far as the numbers are right. Each totally isotropic plane contains twenty-one isotropic points and twenty-one totally isotropic lines. Each totally isotropic line is incident with three totally isotropic planes and each isotropic point lies in twenty-seven totally isotropic planes (recalling the lines in a cubic surface).

Using Edge's picture, but our triangle rather than his XYZ which we describe later, we are led to the following description of the Steiner system S (3, 6, 22) which motivates the search in unitary space:

(2.21) the points are the twenty-two vectors in Λ with twenty-three coordinates equal to one and one coordinate (being neither the first nor the second) equal to minus three, e.g. $(1, 1, -3, 1^{21})$.

(2.22) the blocks are the seventy-seven vectors $(-2, -2, 2^6, 0^{16})$ with -2 in the first two positions and 2 in the other six positions of an eight element $\mathscr{C}$-set.

The vector difference of a point-block pair is of type 3 or 4. A point-block pair is incident if the vector difference of their representatives in Λ is of type 4.

LEMMA 2.23. *In Edge's picture of the projective unitary geometry PU $(6, 2^2)$ inside the Leech lattice the representation of M_{22}* fixes no point.*

Proof. The points of PU $(6, 2^2)$ are lattice points T such that $LMNT$ is a regular tetrahedron with sides of type 2. The vector LT has one of the following forms:

(i) $(2^8, 0^{16})$, (ii) $(3, 1^{23})$, or (iii) $(4^2, 0^{22})$.

If T is of type (i) then the M_{22} acting on the last twenty-two coordinates does not fix the underlying $\mathscr{C}$-set. If T is of type (ii) then MT is of type (i) or (iii) and again if of type (i) the $\mathscr{C}$-set is not fixed by M_{22} but if of type (iii) we need note only that then NT is not of type (ii) but of type (i). Now T cannot be of type (iii) with a 4 in the first two positions; furthermore, any other type (iii) T is moved by M_{22}.

Thus we see that no three planes of set (2.21) are 'copunctual', though any two determine a unique 'point'.

It is consistent with this putative unitary geometry to have twenty-two totally isotropic planes with a common point such that no two have more than this point in common (sixteen of the twenty-seven totally isotropic planes through a given point meet one of the planes in just that point and the remaining ten meet this plane in a line).

Using the above definition of incidence we must show that three points determine a unique block to exhibit a Steiner system S (3, 6, 22). Let $(a_1, \ldots, a_{24})$ be a member of Λ which has vector difference of type 4 with each of $(1, 1, -3, 1, 1, 1^{19})$, $(1, 1, 1, -3, 1, 1^{19})$, $(1, 1, 1, 1, -3, 1^{19})$. A simple calculation using Conway's tabulation of vectors of types 2, 3 and 4 shows that this singles out one vector of Λ and this vector is in (2.22), namely $(-2, -2, 2^6, 0^{16})$ where the underlying octad contains $\{1, 2, 3, 4, 5\}$.

As the group of the lattice Λ induces a group triply transitive on the members of (2.21), we are done.

In terms of unitary geometry this becomes

(2.31) Points: a set of twenty-two totally isotropic planes any two having a point in common, no three having a point in common.

(2.32) Blocks: a set of seventy-seven totally isotropic planes.

(2.33) Incidence: a point is on a block if the corresponding planes have a line in common otherwise the planes are skew.

We have a certain amount of freedom in organizing the search for the required configuration of ninety-nine totally isotropic planes, in particular any given totally isotropic plane π can be assumed to be in set (2.31). Now given a totally isotropic plane which we will refer to as the 'base plane', there are forty-two totally isotropic planes having a line in common with π, ten through each point of π: call this set of planes Ω_{42}. Thus there are $27 - 10 - 1 = 16$ totally isotropic planes having a particular point S and nothing else in common with π. Select one totally isotropic plane π' having only one point S in common with π. An elementary calculation shows that there are, through each of the twenty points of π other than S, just eight totally isotropic planes each of which has but one point in common with π'. Let Ω_{160} denote this set of $20 \cdot 8 = 160$ totally isotropic planes. Once π and π' have been chosen, the candidates for the other twenty points of the Steiner system lie in Ω_{160}.

There are 10 planes of Ω_{42} through $S = \pi \wedge \pi'$. Of these, five have a line in common with π' as is shown by an elementary calculation as before. In fact now each time a plane $\hat{\pi}$ is chosen to represent a point of the Steiner system, 5 planes of Ω_{42} through $\pi \wedge \hat{\pi}$ are forced upon us as blocks. Each time a plane $\beta \in \Omega_{42}$ is designated as a block there will be restrictions on which of the members of Ω_{160} through points of $\beta \wedge \pi$ can be used as points of the Steiner system. In practice these two types of restriction are played off against one another in searching for a representation of the Steiner system.

The above search through Ω_{160} and Ω_{42} yields a total of 22 points and 21 blocks of the putative Steiner system. The remaining 56 blocks are indeed not listed. Rather we proceed to set up a labelling of the points according to Todd's description of the Steiner system and to find matrices which correspond to the three generating permutations of M_{22}. These are then tested to see that they satisfy the relations (4.2) modulo the centre $\{\omega I\}$.

For the purpose of computation, the usual six-dimensional vector space V over F_4 with the Hermitian form $\sum_{i=1}^{6} x_i\bar{x}_i$ is used. Edge remarks that the isotropic points in the corresponding projective space are the one-dimensional subspaces containing only vectors with an even number of non-zero components. It is convenient to work with a basis of isotropic vectors:

$$\begin{aligned} v_1 &= (1,0,0,1,0,0) & v_4 &= (\omega^2,0,0,\omega,0,0) \\ v_2 &= (0,1,0,0,1,0) & v_5 &= (0,\omega^2,0,0,\omega,0) \\ v_3 &= (0,0,1,0,0,1) & v_6 &= (0,0,\omega^2,0,0,\omega) \end{aligned} \tag{2.4}$$

The base plane π is chosen to be

$$x_1 = x_4, \quad x_2 = x_5, \quad x_3 = x_6 \tag{2.51}$$

and the second plane π' to be

$$x_1 = \omega x_4, \quad x_2 = \omega x_5, \quad x_3 = x_6 \tag{2.52}$$

so that $\pi \wedge \pi'$ is $(0, 0, 1, 0, 0, 1) = v_3$. By hand, generating matrices for a subgroup of the unitary group fixing π and π' and transitive on the twenty remaining points of π are computed and stored in the computer together with the coordinates of the eight planes through $(1, 0, 0, 1, 0, 0)$ each having one point in common with π and one point in common with π'. The elements of Ω_{160} are computed. Each of these 160 planes is a three-dimensional subspace of V and the computer is used to find a 160 by 160 matrix whose (i, j) entry is the rank of the subspace generated by $\{\pi_i, \pi_j\}$. We seek a 20 by 20 submatrix of 5's which by the usual arguments of linear algebra corresponds to twenty totally isotropic projective planes, any pair having exactly one point in common.

It is useful to have the machine list the forty-two totally isotropic planes having a line in common with π and to compute 42 by 160 matrix of ranks indexed by $\Omega_{42} \times \Omega_{160}$. We need a submatrix of fours and sixes whose columns are indexed by the twenty points already mentioned. We select the blocks incident with the first element of S (3, 6, 22) simultaneously with the selection of the 22 planes representing the points of S (3, 6, 22). In practice the following lemma is extremely useful in cutting down the size of the search.

LEMMA 2.6. *Assume that the points of the plane π can be made to correspond to totally isotropic planes of unitary six-dimensional geometry over F_4 and that the intersection of planes can be interpreted as in* (2.3). *Then, without loss of generality, in constructing S* (3, 6, 22) *we may start with any configuration of three totally isotropic planes $\{\pi_i, i = 1, 2, 3\}$ such that the three intersections $\pi_i \wedge \pi_j (i \neq j)$ are not collinear.*

Proof. As we are assuming that we can work inside the Leech lattice, it suffices to prove for some triangle XYZ of type 222 that $\cdot 222$ is transitive on triangles of $T_1T_2T_3$ of type 222 such that $XYZT_i$ $(i = 1, 2, 3)$ is of type 222222, using any information about unitary geometry at hand, for example, the easily proven fact that the unitary group is transitive on pairs of totally isotropic planes in a given geometric relation to one another (e.g., a point in common, or a line in common or skew). Thus we may chose the first two points T_1T_2 arbitrarily subject only to the condition that the vector joining these points is of type 2. Using Edge's triangle

$$X = (4, 0, 4, 0^{21}), \quad Y = (0, 4, 4, 0^{21}), \quad Z = (4, 4, 0, 0^{21}),$$

the origin may be taken to be T_1 and the candidates for T_i $(i = 2, 3)$ are of the type $(2, 2, 2, 2^5, 0^{16})$ where an even number of the 2's in 2^5 have a negative sign [**3**]. As T_2 may be chosen arbitrarily from this set, we take $T_2 = (2^8, 0^{16})$ and this requires that $T_3 = (2, 2, 2, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8, 2^4, 0^{12})$ with one or three of the 2's in 2^4 having a negative sign where four of the α_i are zero and the other takes the value -2. Given two positions in this 2^4 there is an involution in M_{24} which fixes each point of the $\mathscr{C}$-set c underlying T_2 and interchanges the given two positions in the 2^4 showing that the positioning of the signs in T_3 is immaterial.

The automorphism group of the Leech lattice contains the reflection that changes all signs on the positions of a $\mathscr{C}$-set but leaves the other positions invariant. Reflection on the complement of c shows that it is immaterial whether there are one or three minus signs in the 2^4 of T_3. Note that this reflection is in $\cdot 222$.

3. The computations for the Steiner system. Matrices, as described above, indexed by $\Omega_{42} \times \Omega_{160}$ and $\Omega_{160} \times \Omega_{160}$ are computed. The set of three planes described in Lemma 2.6 forces the choice of 9 blocks containing the first plane. For each choice of a fourth plane π_4 from Ω_{160} such that $\pi_1 \wedge \pi_i$ $(i = 2, 3, 4)$ are not collinear, three more blocks are determined and the twelve blocks so determined by $\pi_1, \pi_2, \pi_3, \pi_4$ are seen using the cross matrix $\Omega_{42} \times \Omega_{160}$, to extend in eight ways to a set of twenty-one blocks meeting the base plane. We choose among these eight extensions one of which allows the original set of four 'point planes' to be extended to a set of twenty-two planes any two of which have a point in common and no three of which are copunctual.

Having found a candidate for the set of points of S (3, 6, 22) it is tested by constructing involutions fixing the points of a block. The traces of the objects on a block form a hyperovoid of lines. A hyperovoid of points/lines is a set of $n + 2$ points/lines no three of which are collinear/concurrent, whereas an ovoid is a set of $n + 1$ points/lines with this property. By a theorem of Qvist (see e.g. Lüneberg [**8**, p. 61]) a hyperovoid can exist only in projective planes of even order and every ovoid in such a plane extends uniquely to a hyperovoid. The points not on the six lines of the hyperovoid of lines form a hyperovoid of points.

Let S_1, S_2 be two of the six points and σ_1, σ_2 be unitary transvections with centres S_1 and S_2 respectively. Then σ_1 is the required involution of S (3, 6, 22). Enough of these are computed to ensure a group transitive on the 22 planes. Let Σ denote the set of 22 objects. The cross-matrix of ranks $\Sigma^{\sigma_1\sigma_2} \times \Sigma$ is computed. The set Σ is discarded if this matrix is not a matrix of 5's and 3's. Once the objects of S (3, 6, 22) are located they are labelled according to the 'standard' Steiner system (see, e.g. Todd [**9**]). An arbitrary label is assigned to the base plane and, as there is a $1 - 1$ correspondence between points in the base plane and the other 21 objects given by $\pi' \leftrightarrow \pi \wedge \pi'$, one can use the fact that the points of a line are the points of a block containing the base plane. Three labellings arise corresponding to the three orbits of hyperovoids under the little projective group of the plane (see, e.g. Edge [**4**]). One of these is consistent with the standard labelling and this is found by taking a block not containing the base plane and examining the traces of planes corresponding to objects in this block. These traces form a hyperovoid. Each totally isotropic point is either on one of these 77 hyperovoids or is uniquely the intersection of two 'point planes' since $\binom{22}{2} + 77\cdot 6 = 693$.

4. Generators and relations. Explicit generating matrices for $M_{22}{}^*$ are computed. To represent a given permutation P it is convenient to work with the basis of totally isotropic vectors, as given in (2.4) and the planes of (2.5). Now $v_1, v_2, v_3 \in \pi$ and $v_3, v_4, v_5 \in \pi'$ so the images of $v_1, v_2, \ldots, v_5$ can be determined to within scalar multiples, for example, $\lambda v_1 = \pi \wedge \pi^*$ for a uniquely determined π^* among the 22. The image of the subspace generated by v_1 can be located since the images of π and π^* can be found with the aid of P and the Todd labelling. The image of one more object determines these five scalars in terms of one of them. There are two possibilities for the image of v_6 one of which is eliminated via generators and relations.

To obtain generating permutations as well as generators and relations for M_{22} we first use a standard set of defining relations for PSL (2, 11) in terms of the natural generators acting on the projective line over $F_{11} - a : z \to z + 1$, $b : z \to 4z$, $d : z \to -z^{-1}$, and find that $PSL\ (2, 11) = \langle a, b, d\rangle : a^{11} = a^b a^{-3} = b^d b = (ad)^3 = 1$. Writing $d = c^2$ so that $M_{22} = \langle a, b, c\rangle$ and $PSL\ (2, 11) = \langle a, b, c^2\rangle$. We find further relations to obtain

$$M_{22} = \langle a, b, c\rangle : a^{11} = a^b a^{-3} = b^c b^{-2} = (ac^2)^3 = (a^5 c)^5 = (a^3 c)^6 = 1. \tag{4.2}$$

This presentation may be compared with that of Garbe and Mennicke [**5**] and also that found by Todd [**10**] in the course of obtaining his presentation for M_{24}. Coset enumeration enables us to verify the above defining relations and make the identification of $\langle ac, bc\rangle$, which stabilizes coset 17, with PSL (3, 4) also known as M_{21} [**8**, p. 93]. The coset table yields

$a = (1, 2, 4, 8, 16, 9, 18, 13, 3, 6, 12)(5, 10, 20, 17, 11, 22, 21, 19, 15, 7, 14)$

$b = (1, 18, 4, 2, 6)(3)(5, 21, 20, 10, 7)(8, 16, 13, 9, 12)(11, 19, 22, 14, 17)(15)$

$c = (1, 18, 2, 4)(3, 15)(5, 9)(6)(7, 16, 21, 8)(10, 12, 20, 13)(11, 17, 22, 14)(19).$

The permutations a and b are restrictions to 22 symbols of Conway's [**2**, p. 223] β and δ respectively.

Using the methods described above we find unitary matrices A, B, C over $F_4 = \{0, 1, \omega, \omega^2\}$ corresponding to the generators a, b, c respectively.

$$A = \begin{bmatrix} 1 & \omega & 1 & \omega^2 & 0 & \omega^2 \\ 0 & 0 & 0 & \omega^2 & \omega^2 & \omega^2 \\ \omega & 1 & \omega & 0 & \omega & \omega \\ \omega^2 & \omega^2 & 0 & \omega & \omega^2 & 1 \\ 0 & \omega & 0 & \omega^2 & \omega^2 & 0 \\ 0 & \omega & \omega & 1 & \omega & \omega^2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 & \omega^2 & \omega & 0 \\ 0 & \omega^2 & 1 & 1 & \omega^2 & 1 \\ 1 & \omega^2 & 1 & 0 & \omega & \omega^2 \\ \omega & \omega^2 & 1 & 1 & 0 & 1 \\ \omega & \omega^2 & \omega & 0 & \omega^2 & \omega \\ 1 & \omega^2 & \omega^2 & 0 & \omega & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} \omega^2 & \omega & 0 & \omega^2 & 0 & 0 \\ \omega^2 & \omega & \omega & 0 & 1 & \omega \\ \omega & \omega & 0 & \omega^2 & \omega & 1 \\ 1 & 0 & \omega & 1 & \omega & \omega \\ 0 & \omega & 0 & \omega^2 & 1 & 0 \\ 1 & 1 & 1 & \omega^2 & \omega & 0 \end{bmatrix}$$

Conway's γ [**2**, p. 223] generates the outer automorphism of M_{22}; we see that $a^\gamma = a^{-1}$, $b^\gamma = b$, $c^\gamma = c$. From this action and the matrices A, B and C we can deduce the action of the automorphism induced by γ on the centre of M_{22}^*: thus from $a^{11} = \omega$, $b^5 = 1$, $c^4 = \omega$, $(a^5c)^5 = \omega^2$ and $(a^{-5}c)^5 = \omega$ we find that the automorphism must act as $a \to a^{-1}\omega$, $b \to b$ and $c \to c$ and fix the central element of M_{22}^* of order 3. The automorphism is therefore that induced from the transpose map.

5. A representation module for SL(3, 4).

PROPOSITION. *$SL(3, 4)$ has an indecomposable 6-dimensional unitary representation over F_4.*

Proof. It suffices to show that the copy of $SL\ (3, 4) \subset M_{22}^*$ has the unitary space that we have been working with as an indecomposable module.

If the module splits when restricted to the stabilizer of π, then there is a plane π^* which may or may not be totally isotropic such that as a module for $SL\ (3, 4)$ V is the direct sum of these two planes. In either case, π^* contains at least one isotropic point. This point is either on a point plane of the Steiner system in which case one need note only that the stabilizer of a pair of point planes fixes their intersection and is transitive on the remaining points of each to see that this does not arise, or that the isotropic point R in question is on a unique block plane in which case this block plane must intersect π in a line l_ρ. The correspondence $R \to l_\rho$ determines a duality between the two planes π and π^*, and the plane π^* is totally isotropic.

It suffices to show that in the stabiliser of our triangle of type 222 no plane of

$\mathscr{T}$ is fixed by all elements of the stabiliser of $K = (1, 1, -3, 1^{21})$ other than K itself as the members of $\mathscr{T}$ represent the 891 totally isotropic planes. The last 21 coordinates would all have to be equal, hence equal to 0 or 1. If zero, it is $(\pm 4, \pm 4, 0^{22})$. But all vectors in $\mathscr{T}$ are of type 2 and none of these four is in $\mathscr{T}$. If the last 21 coordinates are all one then we have either a vertex or the triangle or K itself. In either case we have a contradiction to the assumption that the module is decomposable.

6. Character tables. For completeness we include a table of the characters of the Schur covering of M_{22}. Degrees are given for characters of M_{22} over a centre of order 2 and of order 3 in [**1**] and Lindsey [**7**, p. 419] is helpful. Gordon James [**6**] has determined the modular characters of the Mathieu groups and we note in passing that the 12-dimensional representation of M_{24} over F_2 described in Conway [**2**, p. 224] contains a 10-dimensional irreducible representation of M_{22} which is a faithful representation of lowest degree for that group.

Each non-zero column of the character table denotes i classes (where i is the order of the centre) with the exceptions of the columns 3 and 4 in $C_6 \cdot M_{22}$ which denote 2 and 3 classes respectively. The orders of the elements are given as for M_{22}.

	1	2	3	4_1	4_2	5	6	7_1	7_2	8	11_1	11_2
	g	384	36	16	32	5	12	7	7	8	11	11
	1	1	1	1	1	1	1	1	1	1	1	1
	21	5	3	1	1	1	−1	0	0	−1	−1	−1
	45	−3	0	1	1	0	0	α	$\bar{\alpha}$	−1	1	1
	45	−3	0	1	1	0	0	$\bar{\alpha}$	α	−1	1	1
M_{22}	55	7	1	−1	3	0	1	−1	−1	1	0	0
	99	3	0	−1	3	−1	0	1	1	−1	0	0
	154	10	1	2	−2	−1	1	0	0	0	0	0
	210	2	3	−2	−2	0	−1	0	0	0	1	1
	231	7	−3	−1	−1	1	1	0	0	−1	0	0
	280	−8	1	0	0	0	1	0	0	0	β	$\bar{\beta}$
	280	−8	1	0	0	0	1	0	0	0	$\bar{\beta}$	β
	385	1	−2	1	1	0	−2	0	0	1	0	0

	1	2	3	4_1	4_2	5	6	7_1	7_2	8	11_1	11_2
	g	384	36	16	32	5	12	7	7	8	11	11
	10	−2	1	0	2	0	1	α	$\bar{\alpha}$	0	−1	−1
	10	−2	1	0	2	0	1	$\bar{\alpha}$	α	0	−1	−1
	56	8	2	0	0	1	2	0	0	0	1	1
	120	8	3	0	0	0	−1	1	1	0	−1	−1
$C_2 \cdot M_{22}$	126	−6	0	0	−2	1	0	0	0	0	β	$\bar{\beta}$
	126	−6	0	0	−2	1	0	0	0	0	$\bar{\beta}$	β
	154	−2	1	0	−2	−1	1	0	0	$2i$	0	0
	154	−2	1	0	−2	−1	1	0	0	$-2i$	0	0
	210	−10	3	0	2	0	−1	0	0	0	1	1
	330	−2	−3	0	2	0	1	1	1	0	0	0
	440	8	−1	0	0	0	−1	−1	−1	0	0	0

	21	5	0	1	1	1	2	0	0	−1	−1	−1
	45	−3	0	1	1	0	0	α	$\bar{\alpha}$	−1	1	1
	45	−3	0	1	1	0	0	$\bar{\alpha}$	α	−1	1	1
	99	3	0	−1	3	−1	0	1	1	−1	0	0
	105	9	0	1	1	0	0	0	0	1	γ	$\bar{\gamma}$
$C_3 \cdot M_{22}$	105	9	0	1	1	0	0	0	0	1	$\bar{\gamma}$	γ
	210	2	0	−2	−2	0	2	0	0	0	1	1
	231	7	0	−1	−1	1	−2	0	0	−1	0	0
	231	−9	0	−1	3	1	0	0	0	1	0	0
	330	−6	0	2	−2	0	0	1	1	0	0	0
	384	0	0	0	0	−1	0	−1	−1	0	−1	−1

	66	6	0	0	2	1	0	α	$\bar{\alpha}$	0	0	0
	66	6	0	0	2	1	0	$\bar{\alpha}$	α	0	0	0
	120	8	0	0	0	0	2	1	1	0	−1	−1
	126	−6	0	0	−2	1	0	0	0	0	β	$\bar{\beta}$
$C_6 \cdot M_{22}$	126	−6	0	0	−2	1	0	0	0	0	$\bar{\beta}$	β
	210	−10	0	0	2	0	2	0	0	0	1	1
	210	6	0	0	−2	0	0	0	0	$2i$	1	1
	210	6	0	0	−2	0	0	0	0	$-2i$	1	1
	330	−2	0	0	2	0	−2	1	1	0	0	0
	384	0	0	0	0	−1	0	−1	−1	0	−1	−1

$\alpha = \frac{1}{2}(-1 + \sqrt{-7}), \beta = \frac{1}{2}(-1 + \sqrt{-11}), \gamma = \frac{1}{2}(1 + \sqrt{-11})$

Acknowledgements. We are grateful to John Conway for suggesting the method for presenting M_{22} and to K. C. Young who carried out much of the computation.

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