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$\mathbb{Z}$-graded identities of the Lie algebras U_1 in characteristic 2

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Abstract

Let K be any field of characteristic two and let U_1 and W_1 be the Lie algebras of the derivations of the algebra of Laurent polynomials $K[t, t^{-1}]$ and of the polynomial ring $K[t]$, respectively. The algebras U_1 and W_1 are equipped with natural $\mathbb{Z}$-gradings. In this paper, we provide bases for the graded identities of U_1 and W_1, and we prove that they do not admit any finite basis.

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1. *Introduction*

The description of the polynomial identities satisfied by an algebra depends heavily on the base field. If the field is of characteristic 0, one may consider multilinear polynomial identities since they determine all identities of a given algebra. One of the main tools in this case is the representation theory of the symmetric and of the general linear groups, and refinements, for a detailed account see the monographs [7, 10]. When the field K is infinite, one has to consider multihomogeneous identities, see for example [7, section 4·2]. The methods one uses in this case are mostly based on Invariant theory [6]. Finally, if K is a finite field then neither of the above types of identities is sufficient. As a rule, neither of the methods described works properly. Instead one uses structure theory [17, 20, 22], together with combinatorics based on the properties of finite fields [24].

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In spite of the extensive research in this area, little is known about the concrete form of the identities satisfied by a given algebra. The monographs [7, 10], and their references, give a good account on the results already obtained. But it should be stressed that the list of algebras whose identities are known is very short and easy to reproduce. In the associative case one knows the identities of the matrix algebras $M_2(K)$ for infinite fields of characteristic different from 2 (see for example [16, 26]). The identities of the Grassmann algebra E are also known, see for example [7, section 5·1], as well as those of its tensor square $E\otimes E$ [25]. Add here the upper triangular matrices of any order n, $UT_n(K)$, and that is it. In the case of Lie algebras one knows the identities of $UT_n(K)$ (as a Lie algebra), and also of $sl_2(K)$ over infinite fields [26, 27]. We only mention that even the identities of $M_3(K)$, over a field of characteristic 0, are not known. A "nagging" and long-standing problem is to determine whether the identities of $M_2(K)$ admit a finite basis whenever K is an infinite field of characteristic 2. It was proved by Kruse and by Lvov, [20, 22] that if R is a finite associative ring then its identities admit a finite basis. Bahturin and Olshansky [3] showed that any finite dimensional Lie algebra over a finite field has a finite basis of identities.

The theory developed by A. Kemer in the eighties [15] led him to the positive solution of the famous Specht problem: is every ideal of identities of an associative algebra in characteristic 0 finitely generated as an ideal of identities? Kemer's results are far from constructive and do not provide concrete bases of the identities of a given algebra.

We have outlined above some of the reasons that led researchers to look for other types of polynomial identities. These include identities with involution, weak identities, group graded identities. We shall not comment on the former two types of identities as we work exclusively with the latter type. Graded identities appeared in Kemer's research; his methods rely heavily on associative superalgebras (that is $\mathbb{Z}_2$-graded algebras). Later on gradings on algebras and their graded identities became an important part of PI theory. The gradings on matrix algebras were described in [4], see also [5], assuming the base field is algebraically closed.

The matrix algebra $M_n(K)$ admits natural gradings by the groups $\mathbb{Z}_n$ and $\mathbb{Z}$. The corresponding graded identities were described in [28, 29] in characteristic 0, and in [1, 2] in positive characteristic. An extensive research on gradings and graded identities for classes of important algebras has been conducted, we direct the interested reader to [18] for further information and references.

The simple finite dimensional Lie algebras over an algebraically closed field of characteristic 0 are well known. In the infinite dimensional case the so-called algebras of Cartan type appear. We denote by W_1 the Lie algebra of derivations of the polynomial ring in one variable $K[t]$, and by U_1 the algebra of derivations of the Laurent polynomials $K[t,t^{-1}]$. The former is known as the Witt algebra, the latter gives rise to the Virasoro algebra. Both algebras have canonical gradings by the group $\mathbb{Z}$ such that every non-zero component is one dimensional. The algebras W_1 and U_1 appear naturally in various branches of Physics and Mathematics, the interested reader can consult the paper [11] and its references for an extensive treatment of this topic.

The n-variable analogues of W_1 and U_1, the algebras W_n and U_n are defined as the derivations of the corresponding polynomial and Laurent polynomial rings in n variables. These were first studied around 1910 by E. Cartan in his classification of simple Lie algebras in characteristic 0. Later on it was discovered that these algebras, although not simple in positive characteristic, produce naturally various simple finite dimensional Lie algebras as their homomorphic images. The celebrated theorems of V. Kac [12, 13] classify the simple Lie algebras graded by $\mathbb{Z}$ under some natural conditions. Namely the algebra $L = \oplus_{i\in\mathbb{Z}} L_i$ must be of polynomial growth: $\sum_{j\le i} \dim L_j$ grows like a polynomial in i; L_0 acts irreducibly on L_{-1}, and L is generated by degrees 0 and ± 1. O. Mathieu [23] classified the simple $\mathbb{Z}$-graded Lie algebras of polynomial growth. The algebras W_n and U_n were also studied by Kaplansky [14].

The graded identities for the algebra W_1 were described in [9], assuming that the field is of characteristic 0. The ones for U_1, with its canonical grading by the group $\mathbb{Z}$ were recently obtained in [8], over an infinite field of characteristic different from two. As a consequence, the main result of [9] was generalised.

In this paper we study the graded identities satisfied by the Lie algebra U_1, equipped with its natural $\mathbb{Z}$-grading, over an arbitrary field of characteristic two. We produce a basis of the graded identities for U_1. As a consequence we obtain a basis of those for W_1 as well. Furthermore, we prove that the graded identities of U_1, as well as these of W_1, do not admit any finite basis. The counterparts of these results over an infinite field of characteristic different from two, were obtained in [8, 9].

The ordinary identities of W_1 coincide with the identities of the Lie algebra of the vector fields on the line if $K = \mathbb{R}$ is the real field. The standard Lie polynomial of order 4 (which is of degree 5 and is alternating in 4 of its variables) is an identity of W_1. On the other hand, it is a long-standing open problem to determine a basis of the identities satisfied by W_1. The vector space of W_n, the derivations of the polynomial ring in n variables, can be given the structure of a left-symmetric algebra, denoted by L_n. In [19] the authors studied the right-operator identities of L_n, and described a large class of general identities for L_n. We hope that our results about the $\mathbb{Z}$-graded identities of U_1 may shed additional light on the polynomial identities satisfied by W_1, and consequently by U_1.

2. *Definitions and preliminary results*

We fix a field K, all algebras and vector spaces we consider will be over K. If A is an associative algebra one defines on the vector space of A the Lie bracket $[a, b] = ab - ba$. Denote by $A^{(-)}$ the Lie algebra thus obtained, the Poincaré–Birkhoff–Witt theorem yields that every Lie algebra is a subalgebra of some $A^{(-)}$.

Let L be an algebra (not necessarily associative) and let G be a group. A G-grading on L is a vector space decomposition

$$\Gamma : L = \oplus_{g\in G} L_g \tag{2·1}$$

such that $L_g L_h \subseteq L_{gh}$, for all $g, h \in G$. In this case one says that L is G-graded. The subspaces L_g are the homogeneous components of the grading and a non-zero element a of L is homogeneous if $a \in L_g$ for some $g \in G$; we denote this by $\|a\|_G = g$ (or simply $\|a\| = g$ when the group G is clear from the context). The support of the grading is the set $\operatorname{supp} L = \{g \in G \mid L_g \neq 0\}$. A subalgebra (an ideal, a subspace) B of A is a graded subalgebra (respectively ideal, subspace) if $B = \oplus_{g\in G}(A_g \cap B)$.

The first example is the Witt algebra $W_1 = Der(K[t])$. It is the Lie algebras of the derivations of the polynomial ring $K[t]$. The elements $e_n = t^{n+1} d/dt$, $n \ge -1$, form a basis of W_1. The Lie algebra structure on the vector space W_1 is given by the multiplication

$$[e_i, e_j] = (j - i)e_{i+j}. \tag{2·2}$$

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When dealing with a field of positive characteristic p, we take the differences $(j-i)$ modulo p.

The algebra W_1 has a $\mathbb{Z}$-grading, $W_1=\oplus_{i\in\mathbb{Z}}L_i$, where $L_i=0$ whenever $i\leq -2$, and L_i is the (one-dimensional) span of e_i if $i\geq -1$. Thus the element e_n is homogeneous of degree n.

Another example of graded algebras is the algebra U_1. Let $A=K[t,t^{-1}]$ be the algebra of Laurent polynomials in one variable t. Then U_1 is the Lie algebras of the derivations of A. The elements $e_n=t^{n+1}d/dt$, $n\in\mathbb{Z}$, form a basis of U_1; the multiplication in U_1 is also given by (2·2). The algebra U_1 is $\mathbb{Z}$-graded, $U_1=\bigoplus_{i\in\mathbb{Z}}L_i$ where L_i is the span of e_i, for each $i\in\mathbb{Z}$. This means that U_1 has full support on $\mathbb{Z}$, in other words $\operatorname{supp} U_1=\mathbb{Z}$.

Since we shall work with the above two graded algebras we will refrain from giving other examples.

Let $X=\cup_{i\in\mathbb{Z}}X_i$ be the disjoint union of infinite countable sets of variables $X_i=\{x_1^i,x_2^i,\ldots\}$, $i\in\mathbb{Z}$. Assuming that for each $i\in\mathbb{Z}$ the elements of the set X_i are of $\mathbb{Z}$-degree i, the free associative algebra $K\langle X_{\mathbb{Z}}\rangle$ has a natural $\mathbb{Z}$-grading $\oplus_{i\in\mathbb{Z}}K\langle X_{\mathbb{Z}}\rangle^i$. Here $K\langle X_{\mathbb{Z}}\rangle^i$ is the vector subspace of $K\langle X_{\mathbb{Z}}\rangle$ spanned by all monomials of $\mathbb{Z}$-degree i. The subalgebra $L\langle X_{\mathbb{Z}}\rangle$ of $K\langle X_{\mathbb{Z}}\rangle^{(-)}$ generated by the set $X_{\mathbb{Z}}$ is the free $\mathbb{Z}$-graded Lie algebra, freely generated by $X_{\mathbb{Z}}$. Note that $L\langle X_{\mathbb{Z}}\rangle$ is a graded subspace of $K\langle X_{\mathbb{Z}}\rangle$ and that the corresponding decomposition gives a $\mathbb{Z}$-grading on $L\langle X_{\mathbb{Z}}\rangle$. The elements of $L\langle X_{\mathbb{Z}}\rangle$ are called $\mathbb{Z}$-graded polynomials (or simply polynomials). The degree of a polynomial f in $x_i^{a_i}$, denoted by $\deg_{x_i^{a_i}} f$ is defined in the usual way. The definitions of multilinear and multihomogeneous polynomials are the natural ones. We define the (left-normed) commutator $[l_1,\ldots,l_n]$ of $n\geq 2$ elements $l_1,\ldots,l_n$ in a Lie algebra L inductively, $[l_1,\ldots,l_n]=[[l_1,\ldots,l_{n-1}],l_n]$ for $n>2$.

Let $L=\oplus_{i\in\mathbb{Z}}L_i$ be a Lie algebra with a $\mathbb{Z}$-grading. An admissible substitution for the polynomial $f(x_1^{a_1},\ldots,x_n^{a_n})$ in L is an n-tuple $(l_1,\ldots,l_n)\in L\times\cdots\times L$ such that $l_i\in L_{a_i}$, for $i=1,\ldots,n$. If $f(l_1,\ldots,l_n)=0$ for every admissible substitution $(l_1,\ldots,l_n)$ we say that $f(x_1^{a_1},\ldots,x_n^{a_n})$ is a graded identity for L. The set of $\mathbb{Z}$-graded polynomial identities of L will be denoted by $T_{\mathbb{Z}}(L)$. It is a $T_{\mathbb{Z}}$-ideal, that is an ideal invariant under the endomorphisms of $L\langle X_{\mathbb{Z}}\rangle$ as a graded algebra. The intersection of a family of $T_{\mathbb{Z}}$-ideals in $L\langle X_{\mathbb{Z}}\rangle$ is a $T_{\mathbb{Z}}$-ideal; given a set of polynomials $S\subseteq L\langle X_{\mathbb{Z}}\rangle$ we denote by $\langle S\rangle_{\mathbb{Z}}$ the intersection of the $T_{\mathbb{Z}}$-ideals of $L\langle X_{\mathbb{Z}}\rangle$ that contain S. We call $\langle S\rangle_{\mathbb{Z}}$ the $T_{\mathbb{Z}}$-ideal generated by S, and refer to S as a basis of this $T_{\mathbb{Z}}$-ideal. It is well known that in characteristic 0, every $T_{\mathbb{Z}}$-ideal $T_{\mathbb{Z}}(L)$ is generated by its multilinear polynomials. Over an infinite field of positive characteristic one has to take into account the multihomogeneous polynomials instead of the multilinear ones.

In this paper, unless otherwise stated, K denotes a field of characteristic two. We do not require any further restrictions on K. Our main result is the following theorem which provides a basis of the $\mathbb{Z}$-graded identities for U_1.

THEOREM 2·1. *Let K be a field of characteristic two. The ideal of the graded identities of U_1 is generated, as a $T_{\mathbb{Z}}$-ideal, by the polynomials*

$$[x_1^a,x_2^b]\equiv 0, \qquad (2\cdot 3)$$

where a and b are integers of the same parity, that is $a\equiv b \pmod 2$.

We deduce, for the graded identities of W_1, the following theorem.

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THEOREM 2·2. *Let K be a field of characteristic two. The $\mathbb{Z}$-graded identities*

$$x^c \equiv 0 \quad (c \leq -2), \qquad [x_1^a, x_2^b] \equiv 0,$$

where a and b are integers greater than -2, and of the same parity, form a basis for the $\mathbb{Z}$-graded identities of the Lie algebra W_1 over K.

3. $\mathbb{Z}$-Graded identities of U_1

Here we prove Theorem 2·1. To this end, we need a series of results.

LEMMA 3·1. *Over a field of characteristic two, the graded identities (2·3) hold for U_1.*

Proof. The proof is immediate by the multiplication rules (2·2).

In analogy with the associative case we will call *monomials* the commutators in the free graded Lie algebra. We recall that a basis for the free Lie algebra $L\langle X_G\rangle$ consists of left-normed monomials, see for more details [21]. Additionally, the Lie algebras U_1 and W_1 are metabelian in characteristic 2 (they are not metabelian otherwise). Thus, without loss of generality, we will consider all monomials to be left-normed.

PROPOSITION 3·2. *Over a field of characteristic two, every graded monomial identity of U_1 is consequence of the identities (2·3).*

Proof. We denote by I the T_G-ideal generated by the polynomials (2·3). We prove the claim by induction on the length n of the monomial. The result is obvious for $n = 2$, so we suppose $n \geq 3$. Let $M' = [x_1^{a_1}, \ldots, x_{n-1}^{a_{n-1}}]$. If $M' \in T_{\mathbb{Z}}(U_1)$ then it lies in I, by the induction hypothesis, and hence $M \in I$. We assume now that $M' \notin T_{\mathbb{Z}}(U_1)$. Let $a = \|M'\|$, then the result of every admissible substitution in M' is a scalar multiple of e_a. Therefore, $M \in T_{\mathbb{Z}}(U_1)$ if and only if $[x_1^a, x_2^{a_n}] \in T_{\mathbb{Z}}(U_1)$. The commutator $[x_1^a, x_2^{a_n}]$ lies in $T_{\mathbb{Z}}(U_1)$ if and only if a and a_n have the same parity, and hence M lies in I. Thus in all cases $M \in I$, as required.

It is well known that every $T_{\mathbb{Z}}$-ideal is generated by its regular polynomials. Recall that a polynomial $f \in L\langle X_{\mathbb{Z}}\rangle$ is regular if every one of its variables appears in every monomial of f, not necessarily with the same degree.

Recall that I is the $T_{\mathbb{Z}}$-ideal generated by the polynomials in (2·3). The next lemma is a key step in the proof of our main theorem.

LEMMA 3·3. *Let $M = [x_{i_0}^{a_0}, x_{i_1}^{a_1}, \ldots, x_{i_n}^{a_n}]$ be a monomial in $L\langle X_{\mathbb{Z}}\rangle$. If $M \notin T_{\mathbb{Z}}(U_1)$ then M, modulo I, can be written in such a way that a_0 is odd, a_i is even for each $i = 1, \ldots, n$, and $a_1 \leq a_2 \leq \cdots \leq a_n$.*

Proof. Since $M \notin T_{\mathbb{Z}}(U_1)$ either a_0 or a_1 is an odd integer. Suppose there exist i and j, with $0 \leq i < j \leq n$, such that a_i and a_j are odd integers. By exchanging the first two variables, if necessary, we can assume that $i = 0$ and $j = n$, and all other a_k are even integers. (If some additional a_k is odd we simply consider the initial part of the commutator, from a_0 to a_k.) Then $M' = [x_{i_0}^{a_0}, x_{i_1}^{a_1}, \ldots, x_{i_{n-1}}^{a_{n-1}}]$ is a monomial in $L\langle X_{\mathbb{Z}}\rangle$ that is not a graded identity. This implies $\sum_{i=0}^{n-1} a_i$ is odd, that is $M \in T_{\mathbb{Z}}(U_1)$ which is absurd. Now we apply identity (2·3), together with the Jacobi identity, several times on the variables of even $\mathbb{Z}$-degree.

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As mentioned earlier, we cannot claim in general that a $T_{\mathbb{Z}}$-ideal is generated by its multihomogeneous elements; this is certainly true if the base field is infinite. But, in our specific case, we are in a position to deduce this fact.

LEMMA 3·4. *Over a field of characteristic two, the $T_{\mathbb{Z}}$-ideal $T_{\mathbb{Z}}(U_1)$ is generated by its multihomogeneous polynomials.*

Proof. The result is well known if K is infinite. So we assume K is a finite field. Let $f=f(x_0,x_1,\ldots,x_n)$ be a regular polynomial in $T_{\mathbb{Z}}(U_1)$. By Lemma 3·3, only one of the variables in f is in an odd component. Suppose x_0 in an odd component and the remaining variables are in even components. Moreover, f is linear in x_0, and x_0 appears in the first position on each monomial in f. We induct on the degree of the polynomial f. Suppose that f is not multihomogeneous. As f is regular, we write $f=\sum_i \alpha_i M_i$ where M_i is the commutator

$$[x_0,x_1,\ldots,x_1,x_2,\ldots,x_{k-1},\underbrace{x_k,\ldots,x_k}_{m_i^k \text{ times}},x_{k+1},\ldots,x_n]=x_0(ad\,x_1)^{m_i^1}\ldots(ad\,x_n)^{m_i^n}.$$

Here m_i^k are non-negative integers depending on the variable x_i and on the monomial M_i, and $ad\,x$ is, as usual, the linear transformation $y\,ad\,x\mapsto[y,x]$ in a Lie algebra. For each variable x_k we put $r_k=\min\{m_i^k\}$. By the regularity of f we have $r_k>0$. Applying identity (2·3), if needed, there exists an element f' which is a sum of regular polynomials: $f'=f'(x_0,x_1,\ldots,x_n)$ in $T_{\mathbb{Z}}(U_1)$, not necessarily having the same variables as f, such that

$$f\equiv f'(ad\,x_1)^{r_1}(ad\,x_2)^{r_2}\cdots(ad\,x_n)^{r_n}\pmod{I}.$$

Note that the degree of the polynomial f' is lower than the degree of f. Since $I\subseteq T_{\mathbb{Z}}(U_1)$, by induction the result follows if $f'\in T_{\mathbb{Z}}(U_1)$. Suppose that $f'\notin T_{\mathbb{Z}}(U_1)$. Then there exists an admissible substitution for the polynomial f, say $(l_0,l_1,\ldots,l_n)$, of elements in U_1 such that $f'(l_0,l_1,\ldots,l_n)\neq 0$, but $f(l_0,l_1,\ldots,l_n)=0$. Recall that $U_1=\oplus_{i\in\mathbb{Z}}L_i$ is the natural $\mathbb{Z}$-grading on U_1. Define A_0 as the sum of all even components L_i, $i\equiv 0\pmod 2$, and A_1 as the sum of the odd ones. Of course, $f'(l_0,l_1,\ldots,l_n)\in A_1$, that is

$$f'(l_0,l_1,\ldots,l_n)=\sum_{i\in\mathbb{Z}}\lambda_i e_{2i+1},$$

where the set $\{i\mid\lambda_i\neq 0\}$ is finite. As each homogeneous component of U_1 is one-dimensional, we have

$$0=f(l_0,l_1,\ldots,l_n)=f'(l_0,l_1,\ldots,l_n)(ad\,e_{a_1})^{r_1}\cdots(ad\,e_{a_n})^{r_n}=\sum_{i\in\mathbb{Z}}\alpha_i\lambda_i e_{(2i+1)+t}.$$

Here $t=\sum_{j=1}^n r_j a_j$. It is clear that each α_i equals 1, this implies that each $\lambda_i=0$, which contradicts the fact that the substitution in f' is not zero. Therefore the graded identity f is equivalent to a multihomogeneous graded identity.

The corollary to the above lemma is not needed in the proof of the main theorem. If K is an infinite field of characteristic different from 2, it was obtained in [8, theorem 4·4]. We mention here that the proof given there is essentially characteristic-free, and also works when K is a finite field. We decided to include it here since, as a rule, the multilinear identities do not determine the ideals of identities if the base field is of positive characteristic.

COROLLARY 3·5. *Let K be a field of characteristic two. The $T_{\mathbb{Z}}$-ideal $T_{\mathbb{Z}}(U_1)$ is generated by its multilinear polynomials.*

Proof. By Lemma 3·4, we can consider the multihomogeneous graded identities. Let $f = f(x_1^{a_1}, \ldots, x_r^{a_r})$ be a multihomogeneous graded identity of U_1. As $\dim_K L_i \leq 1$ for every $i \in \mathbb{Z}$, we have that each admissible substitution φ can be taken to map $x_i^{a_i}$ to $\xi_i e_{a_i}$ where ξ's are commutative and associative (independent) variables over K. Here, as above, e_{a_i} spans the vector space L_{a_i}. Hence we have

$$\varphi(f(x_1^{a_1}, \ldots, x_r^{a_r})) = \xi_1^{n_1} \cdots \xi_r^{n_r} f(e_{a_1}, \ldots, e_{a_r}). \tag{3·1}$$

Fix some a_i, $\deg_{x_i^{a_i}} f = n_i \geq 1$. Take new variables $x_{i,j}^{a_i}$ where $1 \leq j \leq n_i$, and consider a multihomogeneous graded polynomial $h(x_{1,1}^{a_1}, \ldots, x_{1,n_1}^{a_1}, x_{2,1}^{a_2}, \ldots, x_{r,n_r}^{a_r})$ such that (here we put, in order to simplify the notation, $i = 1$)

$$h(\underbrace{x_1^{a_1}, \ldots, x_1^{a_1}}_{n_1}, x_2^{a_2}, \ldots, x_r^{a_r}) = f(x_1^{a_1}, \ldots, x_r^{a_r}).$$

But $x_{i,j}^{a_i}$ can be evaluated to e_{a_i} for each $1 \leq i \leq r$ and $1 \leq j \leq n_i$. Then equation. (3·1) implies that h is a graded identity and that f is a consequence of h. Moreover h is linear in each of the new n_i variables. Continuing in this way for the remaining variables we prove the statement.

Remark 3·6. The choice of the polynomial h in the above theorem need not be given by the multilinearisation process. Due to this reason it need not be unique. For example $f = [x_1^g, x_2^h, x_1^g]$ can come from $g = [x_1^g, x_2^h, x_3^g]$. The complete linearisation of f is $[x_1^g, x_2^h, x_3^g] + [x_3^g, x_2^h, x_1^g]$. Clearly in characteristic 2 we cannot return to f starting with the latter polynomial.

Now we have all the ingredients for the proof of the main result in the paper.

Proof of Theorem 2·1. As done in the proof of Lemma 3·4, every multihomogeneous polynomial f, modulo I, is equivalent, up to a scalar factor, to the monomial

$$[x_0, x_1, \ldots, x_1, x_2, \ldots, x_{k-1}, \underbrace{x_k, \ldots, x_k}_{m_i^k \text{ times}}, x_{k+1}, \ldots, x_n] = x_0(ad\, x_1)^{m_i^1} \cdots (ad\, x_n)^{m_i^n},$$

where $x_1, \ldots, x_n$ are of even degree. Hence $f \in T_{\mathbb{Z}}(U_1)$ if and only if $f \in I$. This implies Theorem 2·1.

The above results are easily adaptable to the case of the Lie algebra W_1. Recall that in [9] the authors considered the base field of characteristic 0, and this was important in their proofs. In [8], the result was extended to W_1 considered over an infinite field of characteristic different from 2.

4. *Independence of graded identities*

In this section we use ideas from [8, 9]. As above K is an arbitrary field of characteristic two.

Denote by $f_{r,s} = [x_1^r, x_2^s] \in L\langle X_{\mathbb{Z}} \rangle$ the graded polynomials from (2·3), and assume $r \leq s$.

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LEMMA 4·1. *Suppose r and s are integers of the same parity,* $r \equiv s \pmod 2$. *The graded identity* $f_{r,s}$ *is not a consequence of the identities* $f_{u,v}$, $u \le v$, *if* $(r, s) \neq (u, v)$.

Proof. Let $r, s \in \mathbb{Z}$ be of the same parity, and let $H = UT(3, K)$ be the Lie algebra of strictly upper triangular 3×3 matrices over K. Define the vector subspaces H_k, $k \in \mathbb{Z}$, in H as follows:

(i) if $r \neq s$ we set $H_k = 0$ for all $k \neq r$, s and $r + s$; H_r is the span of E_{12}, H_s is the span of E_{23} and H_{r+s} is the span of E_{13};

(ii) if $r = s = 0$ then $H_0 = H$, and $H_k = 0$ for every $k \neq 0$;

(iii) if $r = s \neq 0$ then H_r is spanned by E_{12} and E_{23}, H_{2r} is spanned by E_{13}, and $H_k = 0$ for every $k \neq r$, $2r$.

Here E_{ij} is the matrix that has 1 at position (i, j) and 0 elsewhere. It is clear that $H = \oplus_{i\in\mathbb{Z}} H_i$ is a $\mathbb{Z}$-graded Lie algebra. Since $[E_{12}, E_{23}] = E_{13} \neq 0$, the graded identity $[x_1^r, x_2^s]$ is not satisfied in H. On the other hand, one can easily see that H satisfies all graded identities (2·3) as well as all identities $f_{u,v}$ when $(r, s) \neq (u, v)$. The result follows.

A set I of (graded) polynomials is an independent set of (graded) identities if neither of them lies in the ideal of (graded) identities generated by the remaining ones.

COROLLARY 4·2. *The set of polynomials* $\{f_{r,s} \mid r, s \in \mathbb{Z}, r \le s, \quad r \equiv s \pmod 2\}$ *is an independent set of graded identities in* $L\langle X_{\mathbb{Z}}\rangle$.

The above statements, together with Theorem 2·1, yield the following theorem.

THEOREM 4·3. *Over a field of characteristic two, the graded identities*

$$[x_1^a, x_2^b] \equiv 0, \quad a \le b,$$

where a and b are of the same parity, form a minimal basis for the $\mathbb{Z}$*-graded identities of* U_1.

For the Lie algebra W_1, we add to the list of identities the variables x^c with $c < -1$. Note that the identity $f_{-1,-1} = [x_1^{-1}, x_2^{-1}]$ is a consequence of x^{-2}.

LEMMA 4·4. *For each* $d \in \mathbb{Z}$, *the graded identity* x^d *is not a consequence of the graded identities* (2·3) *and all identities* x^c *where* $c \neq d$.

Proof. Let $d \in \mathbb{Z}$ and let H be the 1-dimensional Lie algebra over K. The algebra $H = \oplus_{i\mathbb{Z}} H_i$ is $\mathbb{Z}$-graded with ith homogeneous component H_i equal to H if $i = d$ and 0 otherwise. It is clear that H satisfies the graded identities (2·3) as well as all graded identities x^c where $c \neq d$ but does not satisfy the identity x^d.

COROLLARY 4·5. *The set of polynomials* $\{x_d \mid d \le -2\}$ *is an independent set of graded identities in* $L\langle X_{\mathbb{Z}}\rangle$.

THEOREM 4·6. *The graded identities* x^c $(c \le -2)$ *and* $[x_1^a, x_2^b]$, *where a and b are of the same parity, with* $0 \le a \le b$, *form a minimal basis for the* $\mathbb{Z}$*-graded identities of* W_1 *over a field K of characteristic* 2.

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The next corollary is a direct consequence of the above theorem together with Theorem 4·3.

COROLLARY 4·7. *Over a field of characteristic two, the $\mathbb{Z}$-graded identities for the Lie algebra U_1, as well as W_1, do not admit any finite bases.*

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