STABILITY AND BIFURCATIONS OF SYMMETRIC PERIODIC ORBITS IN THE RESTRICTED 3-BODY PROBLEM

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ABSTRACT. The continuation of symmetric periodic orbits can be described in terms of "symmetry functions"; the branching of the zero-level lines in a neighbourhood of a critical point gives rise to the transition from "first kind" to "second kind" periodic orbits. When the families are parametrized with the Jacobi integral, the bifurcations can be described as "catastrophes" of the generating functions. However bifurcations of higher order are more complex than the generic catastrophes with one parameter: both symmetric and asymmetric bifurcations occur.

In this way the symmetric periodic orbits that do not have close approaches to the secondary body can be described in an analytic way and their stability can be deduced from simple bifurcation rules. However numerical experiments are required to determine the "natural termination" of the families.

1. CONTINUATION OF SYMMETRIC PERIODIC ORBITS

The "synodic" two-body problem can be described by the hamiltonian:

$$H_o = -m_1^2/2\Lambda^2 - \Lambda + (\eta^2+\xi^2)/2 \tag{1}$$

in the "synodic Poincaré variables", which are defined in terms of the usual keplerian elements a,e,ω,M of the orbit around m_1, by:

$$\begin{aligned}
\lambda &= M+\omega-t \\
\eta &= [(4m_1a)^{\frac{1}{2}}(1-(1-e^2)^{\frac{1}{2}}]^{\frac{1}{2}}\cos(\omega-t) \\
\Lambda &= (m_1a)^{\frac{1}{2}} \\
\xi &= [(4m_1a)^{\frac{1}{2}}(1-(1-e^2)^{\frac{1}{2}}]^{\frac{1}{2}}\sin(\omega-t)
\end{aligned} \tag{2}$$

For $e=0$, λ is defined as the angle formed by the two bodies and the rotating x axis; in this way $(\lambda,\eta,\Lambda,\xi)$ are defined for negative energy and positive angular momentum of the osculating orbit.

We will deal with "symmetric perturbations" to H_o, i.e. hamiltonian problems of the form

V. V. Markellos and Y. Kozai (eds.), Dynamical Trapping and Evolution in the Solar System, 225–233.

https://doi.org/10.1017/S0252921100097116 Published online by Cambridge University Press

$$H=H_o+R(\lambda,\eta,\Lambda,\xi,\mu) \tag{3}$$

such that R=0 for $\mu=0$, and the involutive transformation

$$\sigma:(\lambda,\eta,\Lambda,\xi)\longrightarrow(-\lambda,\eta,\Lambda,-\xi) \tag{4}$$

leaves R invariant. Every perturbing function R depending only on the "synodic" coordinates x,y and even in y has this property; e.g. the perturbing function of the restricted 3-body problem in synodic eliocentric coordinates x,y:

$$R=\mu[x-1/((x-1)^2+y^2)^{\frac{1}{2}}] \qquad \mu=1-m_1 \tag{5}$$

We will anyway suppose that T is real-analytic apart from a finite number of singularities for each value of μ.

As a corollary of the "mirror theorem" (Roy and Oveden,1955), if on an orbit of H there are two symmetric configurations (i.e. $\lambda=0$ or π, $\xi=0$) at two different times, e.g. t=0 and t=T, the orbit is periodic of period 2T and symmetric with respect to σ.

This "symmetry condition" can be computed using the "surface of section" method: chosen an hypersurface $S_\ell:\lambda=\ell\pi$ as a cross section ($\ell=0,1$), we call the first crossing of $S_{\ell+s}$ the "Poincaré map of order s" (s an integer $\neq$ 0):

$$\theta_{\ell,s}:(\eta,\Lambda,\xi)\longrightarrow(\eta_1,\Lambda_1,\xi_1) \tag{6}$$

then we impose $\xi_1=0$ for $\xi=0$, and we have a "symmetry function" $\Phi_{\ell,s}(\eta,\Lambda,\mu)=0$ defining for every μ the "characteristic lines" in the η, Λ plane, corresponding to symmetric periodic orbits of order s (s synodic revolutions in a period).

For $\mu=0$ the symmetric periodic orbits are easily computed because the general integral of (1) can be described as $\Lambda=\Lambda_o$, $\lambda=n(\Lambda_o)t+\lambda_o$, and (η,ξ) rotating clockwise with angular velocity 1; where $n(\Lambda)=m_1^2/\Lambda^3$ is the usual mean motion; then:

$$\Phi_{\ell,s}(\eta,\Lambda,0)=-\eta\ \sin[s\pi/(n(\Lambda)-1)]=0 \tag{7}$$

has the solutions $\eta=0$, $\xi=0$ (circular orbits) and $\xi=0$, $n(\Lambda)=(s+k)/k$, k an ineteger $\neq$ 0 (resonant orbits of order s).

In this setting we can state a "continuation theorem" which gives more informations than the classical ones (Poincaré,1892; Arenstorf, 1963; Barrar,1965). We say that a subset W of the η,Λ plane in S_ℓ is "s-safe" if every point $(\ell\pi,\eta,\Lambda,0)$ with $(\eta,\Lambda)\epsilon W$ lies on an orbit that for $\mu=0$ does not hit a singular point of R (lying on $\mu=0$) before it crosses $S_{\ell+s}$. Note that the singular points of R, as an analytic function, lying on $\mu=0$, do exist even if $R\equiv0$ on $\mu=0$; e.g. the restricted 3-body problem for $\mu=0$ is not the two-body problem, but the two-body problem with collisions (Brjuno,1978).

Continuation theorem: let W be an open subset of the η,Λ plane in

S_ℓ, such that its closure $U \cup \partial W$ is compact, s-safe, contained in $n(\Lambda)\neq 1$, $\eta^2<2\Lambda$; suppose that there an no circular resonant orbits of order s (i.e. points with $\mu=0$, $n(\Lambda)=(s+k)/k$, k integer $\neq 0$) on the boundary ∂W.

Then for $|\mu|<\mu_1$, μ_1 depending on W,s,ℓ only, the symmetry function $\Phi_{\ell,s}(\eta,\Lambda,\mu)$ is a Morse function of η,Λ on W (i.e. it has only a finite number of nondegenerate critical points); moreover all these critical points are of saddle type, and are "near" the circular resonant orbits of order s.

Proof.: $n(\Lambda)\neq 1$ and s-safety ensure the smoothness of the symmetry function; (7) shows that for $\mu=0$ it has only critical points of saddle type, and Morse functions are "stable".

However, the qualitative behaviour of the characteristics is determined not only by the location of the critical points, but also from the value of the symmetry function at the critical points:

Shallow resonance lemma: let Λ_o be a resonant value (of order s), i.e. $n(\Lambda_o)=(s+k)/k$, k an integer $\neq 0$, and let the first order perturbation to the symmetry function at the corresponding circular orbit be nonzero:

$$\frac{\partial}{\partial\mu}\Phi_{\ell,s}(0,\Lambda_o,0)\neq 0 \tag{8}$$

Then the characteristics of order s are smooth curves in a neighbourhood of $(0,\Lambda_o)$ for $0<\mu<\mu_2$, μ_2 depending on s,ℓ,k only.

Proof.: In the Taylor expansion of the critical value of the symmetry function in powers of μ, (8) turns out to be the first order term. Therefore for μ small enough and $\neq 0$ the critical point does not lie on the characteristic.

"Shallow resonance" refers to the hyperbola-like branches that connect the periodic orbits of the "first kind" (continuation of the $\eta=0$ branch) with those of the "second kind" (continuation of the $n(\Lambda)=$ $=(s+k)/k$ branch). This "quadratic approximation" was discussed by Guillaume (1969), who also computed (in a different coordinate system) the first-order perturbation to the symmetry function for the restricted 3-body problem and for s=1, obtaining a formula equivalent to:

$$n(\Lambda_o)=\frac{k+1}{k}\Rightarrow\frac{\partial\Phi_{\ell,1}}{\partial\mu}(0,\Lambda_o,0)\cdot(-1)^{k+\ell}\frac{k}{|k|}>0 \tag{9}$$

and this is enough to understand the structure of shallow resonances;in figure 1 we adapted from Colombo et al.(1969) the plot of the characteristic lines for $\mu\simeq 1/1048$ ("asteroidal problem"); the plot refers to the a, $\tilde{e}=e\cos(\omega-t)$ plane instead of the Λ,η plane, but the qualitative features of the characteristics are the same, and are much like the theory although μ is not "very small".

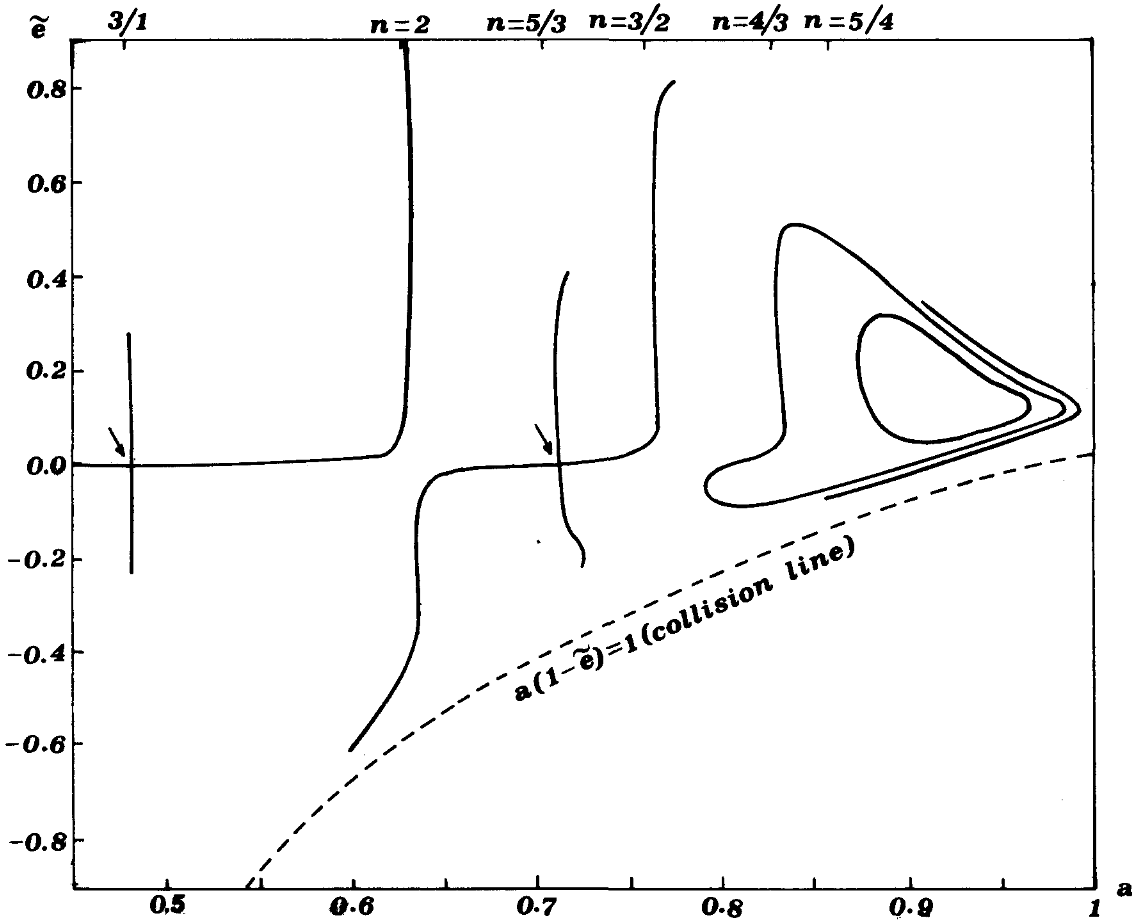

Figure 1

Symmetric periodic orbits on the a, $\tilde{e}=e\cos(\omega-t)$ plane; $\lambda=0$. n=2, 3/2, 4/3 and 5/4 families from Colombo et al. (1969), n=3 and 5/3 from numerical experiments performed by the author.

https://doi.org/10.1017/S0252921100097116 Published online by Cambridge University Press

For order $s>1$, the situation is different:

Deep resonance lemma: let Λ_o be a resonant value of order s strictly > 1, i.e. $n(\Lambda_o)=(s+k)/k$, k integer but $n(\Lambda_o)\neq(1+k)/k$, k any integer. Then there is no shallow resonance, i.e. 0 is a critical value corresponding to a critical point near $(0,\Lambda_o)$ for the symmetry function and the two branches corresponding to first and second kind periodic orbits cross transversally, for $0<\mu<\mu_3$, μ_3 depending only on k,s.

Proof.: If (η,Λ) defines a symmetric periodic orbit of order 1, then it defines also a symmetric periodic orbit of order s, for every s integer $\neq 0$; since the characteristic of order 1 is smooth, far from circular resonances of order 1, we conclude that the symmetry function of order 1 divides the symmetry function of order s, the quotient being a smooth function with zero level corresponding to the second kind branch.

2. STABILITY AND BIFURCATIONS

Since the Hamiltonian $H=H_o+R$ is an integral, the Poincaré map (6) can be "reduced" to a mapping of a two-dimensional manifold in itself; this can be done by solving for Λ in $H=h$, provided $H_\Lambda=\dot{\lambda}\neq 0$. In this way we can define a map, depending on two parameters h,η:

$$T^{h,\mu}_{\ell,s}:(\eta,\xi)\longrightarrow(q,p)=(Q(\eta,\xi),P(\eta,\xi)) \tag{10}$$

with $q=\eta_1$, $\xi=p_1$; if s is even, T maps S_ℓ in itself, and periodic orbits of order s/2 appear as fixed points. The map T is symplectic, or area-preserving, because λ and Λ are conjugate variables (Siegel and Moser, 1971, §22): its jacobian matrix A satisfies:

$$\det A=Q_\eta P_\xi-Q_\xi P_\eta=1 \tag{11}$$

and it can be (locally) defined with a generating function; if T is not too far from identity, such that $Q_\eta>0$, $P_\xi>0$, we can use a generating function $\mathcal{S}(\xi,q)=-\xi q-S(\xi,q)$ to define (10) by:

$$\eta=q+S_\xi(\xi,q) \qquad p=\xi+S_q(\xi,q)\ ; \tag{12}$$

then the jacobian matrix $A=A(\eta,\xi,h,\mu)$ of (10) can be computed as a function of the derivatives of S:

$$\begin{aligned} Q_\eta&=(1+S_{\xi q})^{-1} & Q_\xi&=-S_{\xi\xi}(1+S_{\xi q})^{-1} \\ P_\eta&=S_{qq}(1+S_{\xi q})^{-1} & P_\xi&=1+S_{q\xi}-S_{qq}S_{\xi\xi}(1+S_{\xi q})^{-1} \end{aligned} \tag{13}$$

To extract stability informations from this canonical formalism we use the Arnold (1976, appendix 9) method, defining an auxiliary function $F(\eta,\xi)=S(\xi,Q(\eta,\xi))$; all the critical points $F_\eta=F_\xi=0$ of the function F are fixed points of the map (10), (12); to investigate the linearized map A at the fixed points, we can compute the hessian matrix B of F at the critical points:

$$F_{\eta\eta}=S_{qq}Q_\eta^2 \qquad F_{\eta\xi}=(S_{q\xi}+S_{qq}Q_\xi)Q_\eta$$
$$F_{\xi\xi}=S_{\xi\xi}+S_{\xi q}Q_\xi+(S_{q\xi}+S_{qq}Q_\xi)Q_\xi \tag{14}$$

and by using (13), (14) and (11) we get B as a function of A:

$$B=\begin{bmatrix} Q_\eta P_\eta & Q_\eta(P_\xi-1) \\ Q_\eta(P_\xi-1) & Q_\xi(P_\xi-2) \end{bmatrix} \tag{15}$$

We will say that a fixed point of T is <u>linearly stable</u> whenever its characteristic multipliers (eigenvalues of A) are complex numbers with absolute value 1, different from ± 1; that it is <u>linearly unstable</u> whenever the multipliers are real numbers ν, $1/\nu$ <u>different from ± 1</u>; linearly critical whenever the multipliers are (twice) +1 or -1. I remember that a linearly unstable periodic orbit is unstable in the ordinary sense, by the Hartmann and Grobman theorem (Hartmann,1964); in the linearly stable case more information is needed to apply KAM theory.

<u>Critical point theorem</u>: let the map (10) (with $Q_\eta>0$, $P_\xi>0$) be represented by (12); then the auxiliary function F has saddles corresponding to linearly unstable fixed points of T, extrema corresponding to linearly stable ones, and degenerate critical points corresponding to linearly critical fixed points.

<u>Proof.</u>: By (15) and (11):

$$\det B=Q_\eta(2-Q_\eta-P_\xi)=Q_\eta[2-\mathrm{Tr}\ A]$$

and because the multipliers satisfy the equation:

$$\nu^2-\mathrm{Tr}\ A\nu+1 = 0$$

det $B \gtrless 0$ correspond to complex or real multipliers.

To investigate the linearly critical case, we use also the so-called "<u>Henon stability criterium</u>": if the fixed point of T is on the symmetry line $\xi=0$, then $Q_\eta=P_\xi$, and linear stability corresponds to $Q_\eta=P_\xi<1$. Then the linearly critical case for a symmetric periodic orbit is characterized by $P_\eta=0$ or $Q_\xi=0$; we will say that a linearly critical, symmetric orbit is undergoing a <u>symmetric degeneracy</u> whenever $Q_\eta=P_\xi=1$, $P_\eta=0$; i.e. whenever the eigenspace with eigenvalue 1 contains the symmetry axis $\xi=0$.

https://doi.org/10.1017/S0252921100097116 Published online by Cambridge University Press

Symmetric bifurcation theorem: suppose that the number of symmetric periodic orbits of order s relative to the cross section $S_\ell(\ell=0,1)$, lying on the level surface H=h, changes at the value h_o near the point (η_o,Λ_o) (i.e. this change occurs regardless of the neighbourhood of (η_o,Λ_o) to which we restrict the count). Then (η_o,Λ_o) corresponds to a symmetric periodic orbit of order s, relative to S_ℓ, satisfying each of the following equivalent properties:

(A) the periodic orbit through $(\ell\pi,\eta_o,\Lambda_o,0)$ is undergoing symmetric degeneracy: $Q_\eta=P_\xi=1$, $P_\eta=0$.

(B) the auxiliary function F (can be defined for h near h_o),and has a degenerate critical point at $(\eta_o,0)$ with hessian matrix:

$$B=\begin{bmatrix} 0 & 0 \\ 0 & -Q_\xi \end{bmatrix} \tag{16}$$

(C) either the level lines of H in the η,Λ plane are tangent at (η_o,Λ_o) to the characteristic line $\Phi_{\ell,s}(\eta,\Lambda,\mu)=0$, or the characteristic line itself has a singularity at (η_o,Λ_o).

Proof.: Since the symmetry function is real-analytic, a symmetric periodic orbit cannot disappear, but only bifurcate: this requires det (A-Id)=0; moreover the eigenspace with eigenvalue 1 of A must contain the direction from which the bifurcating fixed points approach,i.e. (A). Then F can be defined, and (15) gives (B). Let us compute P_η as the total derivative of ξ_1 with respect to η on $H=h_o$:

$$P_\eta = \frac{\partial\xi_1}{\partial\eta} - \frac{\partial\xi_1}{\partial\Lambda}\,\frac{\partial H}{\partial\eta}\left(\frac{\partial H}{\partial\Lambda}\right)^{-1} \tag{17}$$

and $P_\eta=0$ implies (C).

A kind of inverse statement allowing an explicit description of a generic bifurcation is the following:

Fold catastrophe theorem: let for $H=h_o$ there be a symmetric periodic orbit (η_o,Λ_o) of order s, relative to S_ℓ, with Q_η, $P_\xi>0$; suppose that the characteristic $\Phi_{\ell,s}(\eta,\Lambda,\mu)$ is smooth at (η_o,Λ_o) and that the level line $H=h_o$ has a contact of order 2 with it at (η_o,Λ_o), i.e. they have the same tangent but a different curvature, in such a way that they do not cross each other (in a neighbourhood of (η_o,Λ_o)); suppose also that there is no asymmetric degeneracy, i.e. $Q_\xi\neq0$ in a neighbourhood. Then:

(A) the number of symmetric periodic orbits of order s, relative to S_ℓ, lying on H=h changes at the value h_o near (η_o,Λ_o): two symmetric periodic orbits, one linearly stable and one linearly unstable, collide at (η_o,Λ_o) and disappear.

(B) for $h=h_o$ the auxiliary function F has a degenerate critical point at $(\eta_o,0)$ with hessian matrix B as in (16), with the third derivative

$F_{\eta\eta\eta}(\eta_o,0)\neq 0$; for the parameter h varying across h_o, F undergoes a "fold catastrophe", i.e. an extremum collides with a saddle and disappears.
(C) the periodic orbit through $(\ell\pi,\eta_o,\Lambda_o,0)$ undergoes a symmetric degeneracy, i.e. for $h=h_o$ $P_\eta(\eta_o,0)=0$, with the second derivative

$P_{\eta\eta}(\eta_o,0)\neq 0$.

Proof.:The condition on the curvatures gives $P_{\eta\eta}\neq 0$;i.e.(C);by (15), $F_{\eta\eta\eta}=Q_\eta P_{\eta\eta}+Q_{\eta\eta}P_\eta$ and because of the degeneracy $P_\eta=0$, $P_{\eta\eta}\neq 0$ implies $F_{\eta\eta\eta}\neq 0$; the sign of $F_{\eta\eta}$ changes in the prescribed way, giving (B). Then (A) follows, by the critical point theorem.

In this way, since first kind periodic orbits are known to be linearly stable outside the resonances of order 1,2, if there are no "asymmetric" degeneracies and bifurcations the linear stability character of the symmetric periodic orbits can be determined by only plotting the characteristic lines and the level lines of H in the η,Λ plane.

Higher codimension catastrophes are not to be expected "generically", since the bifurcations depend on only one parameter h;however, this applies strictly only to order s=1, because the auxiliary functions F that are related to symplectic map squared are not an open set in the space of smooth functions. Therefore the bifurcations of higher order resonant orbits from "first kind" periodic orbits are more complex. We will now discuss the case of order s=2.

Even order bifurcation theorem: let s be even and s/2 be odd, and let $\eta=0$, $\Lambda=\Lambda_o$ correspond to a circular orbit resonant of order s (and not of order $<$ s); let $h_o=H(\ell\pi,0,\Lambda_o,0,0)$. Let P, Q be the components of the map $T^{h_o,\mu}_{\ell,2s}$ and let their first order perturbations satisfy:

$$\left.\frac{\partial P_\eta}{\partial \mu}\right|_{\mu=0}+\left.\frac{\partial Q_\xi}{\partial \mu}\right|_{\mu=0}\neq 0 \tag{18}$$

then for μ small enough and $\neq 0$ the family of "first kind" symmetric periodic orbits (continuation of the $\ell=0$ branch) undergoes two separate bifurcations, one symmetric and one asymmetric. However the asymmetric bifurcation for $\ell=0$ is the symmetric one for $\ell=1$ and viceversa.

Proof.: For $\mu=0$ let us plot the lines $P_\eta=0$, $Q_\xi=0$ on the (η,Λ) plane:

$$P_\eta=-\sin t+O(\eta^2) \qquad Q_\xi=\sin t+O(\eta^2) \qquad t=2\pi s/(n(\Lambda)-1) \tag{19}$$

therefore (18) ensures that the two curves $P_\eta=0$ and $Q_\xi=0$ cross transversally the $\eta=0$ line in two distinct point for $\mu\neq 0$ small enough; they also must cross the "first kind" branch in two distinct points,giving one symmetric and one asymmetric degeneracy. By the "deep resonance lemma" the symmetric degenerate periodic orbit must be the crossing point between the first kind and the second kind periodic orbits. On the other hand the map $T^{h_o,0}_{\ell,s/2}$ has a linear part of the form $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ at the resonance, therefore the asymmetric degeneracies on $\lambda=0$ are transformed into symmetric degeneracies on $\lambda=\pi s/2$ and viceversa for $\mu=0$; $h=h_o$:

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & 0 \\ c & a \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & -c \\ 0 & a \end{bmatrix}$$

Because of the Henon property $Q_\eta=P_\xi$, the eigenspace of eigenvalue -1 for the derivative of $T^{h,\mu}_{1,s}$ must be either the η axis or the ξ axis, therefore it is the η axis for μ near 0; this shows that $Q_\xi=0$ on $\lambda=0$ corresponds to $P_\eta=0$ on $\lambda=\pi$ for μ small enough: this is the point of symmetric bifurcation at the opposition.

The study of the auxiliary function F for $T^{h,\mu}_{\ell,2s}$ shows that the stability of the bifurcating second kind orbits is determined also by the sign of the expression (18): e.g. in the interior case $\mathbf{n}(\Lambda)>1$, the bifurcation occurring for smaller Λ gives rise to stable second kind orbits, that occurring for larger Λ gives rise to unstable second kind orbits. Numerical experiments easily show that, in the restricted 3-body problem, for s=2 condition (18) holds and moreover the sign is such that in the interior case the stable resonant orbits of order 2 are symmetric at the oppositions (since pericenter and apocenter at the conjunction gives rise to close approach).

In fig. 1 we have plotted not only the orbits of order 1 found by Colombo et al. (1969), but also the n=3/1 and the n=5/3 families;since fig.1 refers to conjunctions, the asymmetric bifurcations occur "before" the symmetric ones, and they are marked with arrows.

3. REFERENCES

Arnold,V.:1976,*Methodes mathematiques de la mécanique classique*, MIR, Moscow.

Arenstorf,R.F.:1963, Am.J.Math. 85, p.27.

Barrar,R.B.:1965, Astron.J. 70, p.3.

Brjuno,A.D.:1978, Celestial Mechanics 18, p.9.

Colombo,G. et al.:1968, Astron.J. 73, p.111.

Guillame,P.:1969, Astron.Astrophys. 3, p.57.

Hartmann,P.:1964, *Ordinary differential equations*, J.Wiley & Sons.

Poincaré,H.:1892, *Les méthodes nouvelles de la Mécanique Céleste*,vol.I, Gauthier-Villars.

Roy,A.E.,Ovenden,M.W.:1955, Mon.Nat.R.Astron.Soc. 115, p.297.

Siegel,C.L., Moser,J.K.:1971, *Lectures on Celestial Mechanics*,Springer.